

ON LOCAL AND GLOBAL RIGIDITY OF QUASI-CONFORMAL ANOSOV DIFFEOMORPHISMS

BORIS KALININ AND VICTORIA SADOVSKAYA

*Department of Mathematics, University of Michigan, Ann Arbor,
MI 48109-1109, USA* (kalinin@umich.edu; sadovska@umich.edu)

(Received 23 July 2002; accepted 10 December 2002)

Abstract We consider a transitive uniformly quasi-conformal Anosov diffeomorphism f of a compact manifold \mathcal{M} . We prove that if the stable and unstable distributions have dimensions greater than two, then f is C^∞ conjugate to an affine Anosov automorphism of a finite factor of a torus. If the dimensions are at least two, the same conclusion holds under the additional assumption that \mathcal{M} is an infranilmanifold. We also describe necessary and sufficient conditions for smoothness of conjugacy between such a diffeomorphism and a small perturbation.

Keywords: rigidity; Anosov systems; conformal structures; smooth conjugacy

AMS 2000 *Mathematics subject classification:* Primary 37C; 37D

1. Introduction

The goal of this paper is to investigate the local and global rigidity of uniformly quasi-conformal Anosov diffeomorphisms. First we study the global rigidity, i.e. a classification of these systems up to C^∞ conjugacy. Then we consider local rigidity, i.e. the question when an Anosov diffeomorphism is smoothly conjugate to a small perturbation.

The global rigidity properties of conformal and quasi-conformal Anosov systems were studied by Sullivan [18], Kanai [13] and Yue [19] in the case of geodesic flows of compact manifolds of negative curvature of dimension at least three. Based on the earlier work of Kanai, Yue showed that if the flow is uniformly quasi-conformal, then the manifold is of constant curvature. The approach used in these papers was centred at studying the properties of the sphere at infinity. In [17], Sadovskaya considered a more general class of uniformly quasi-conformal contact Anosov flows and symplectic Anosov diffeomorphisms. Using a different approach, she proved that these systems are essentially C^∞ conjugate to algebraic models, i.e. automorphisms of tori and geodesic flows of compact manifolds of constant negative curvature. The following theorem generalizes this result for the case of arbitrary transitive uniformly quasi-conformal Anosov diffeomorphisms.

Theorem 1.1. *Let f be a transitive C^∞ Anosov diffeomorphism of a compact manifold \mathcal{M} which is uniformly quasi-conformal on the stable and on the unstable distributions. Suppose either that both distributions have dimension at least three, or that they have*

dimension at least two and \mathcal{M} is an infranilmanifold. Then f is C^∞ conjugate to an affine Anosov automorphism of a finite factor of a torus.

The example in §4 shows that there exist non-trivial finite factors of conformal Anosov toral automorphisms. We prove this theorem in §3. We use the results of Sadovskaya in [17] and the theorem of Benoist and Labourie in [1] on classification of Anosov diffeomorphisms with C^∞ Anosov splitting preserving a C^∞ affine connection.

We note that the global rigidity results described above do not have analogues for systems with one-dimensional stable or unstable distributions, since all maps in dimension one are conformal.

We apply Theorem 1.1 to investigate the local rigidity of quasi-conformal Anosov diffeomorphisms. The proofs of all the statements are given in §5.

If f is an Anosov diffeomorphism and g is sufficiently C^1 close to f , then it is well known that f and g are topologically conjugate. In general, the conjugacy is only Hölder continuous. A necessary condition for the conjugacy to be C^1 is that the Jordan normal forms of the derivatives of the return maps of f and g at the corresponding periodic points are the same. This condition is also sufficient in the case of Anosov systems with one-dimensional stable and unstable distributions [2, 3, 6, 16]. Examples constructed by de la Llave in [3] show that this condition, in general, is not sufficient for higher-dimensional systems. In contrast to the one-dimensional case, these examples have different stable (unstable) Lyapunov exponents, i.e. different exponential rates of contraction (expansion). However, even if there is only one stable and one unstable Lyapunov exponent, one cannot expect to generalize the one-dimensional result. Indeed, in [4], de la Llave gave an example of an automorphism of a four-dimensional torus with double stable (unstable) eigenvalue and a non-trivial Jordan block in the stable (unstable) direction, for which the condition above does not guarantee C^1 conjugacy to a perturbation. This suggests that the quasi-conformality of the unperturbed system is a natural assumption for higher-dimensional generalizations. Indeed, for a uniformly quasi-conformal Anosov diffeomorphism, the derivative of the return map at any periodic point is diagonalizable over \mathbb{C} , and all of its stable (unstable) eigenvalues are equal in modulus. In [4], de la Llave proved that the coincidence of the Jordan normal forms at the corresponding periodic points guarantees the smoothness of the conjugacy for a certain class of conformal Anosov system. We discuss this result in more detail below. If the unperturbed system is uniformly quasi-conformal, the following corollary of Theorem 1.1 gives alternative necessary and sufficient conditions for the regularity of the conjugacy.

Corollary 1.2. *Let f be a diffeomorphism as in Theorem 1.1, and let g be a C^∞ Anosov diffeomorphism of \mathcal{M} that is topologically conjugate to f . Then uniform quasi-conformality of g is necessary for the conjugacy to be Lipschitz and sufficient for the conjugacy to be C^∞ .*

Next we would like to describe some conditions on the derivatives of the perturbed system at the periodic points which guarantee its quasi-conformality. We use the following proposition, which can be viewed as an analogue of a non-commutative Livšic theorem.

Proposition 1.3. *Let g be a transitive Anosov diffeomorphism of a compact manifold \mathcal{M} satisfying the following ‘bunching’ conditions: there exist numbers $0 < \nu_s \leq \mu_s < 1 < \mu_u \leq \nu_u$ such that, for all $x \in \mathcal{M}$,*

$$\begin{aligned} \nu_s \|v\| &\leq \|df_x^s(v)\| \leq \mu_s \|v\| && \text{for all } v \in E^s(x), \\ \mu_u \|v\| &\leq \|df_x^u(v)\| \leq \nu_u \|v\| && \text{for all } v \in E^u(x), \end{aligned}$$

and

$$\frac{\mu_s^2}{\nu_s} < 1, \quad \frac{\mu_s}{\nu_s \mu_u} < 1, \quad \frac{\nu_u}{\mu_u^2} < 1, \quad \frac{\nu_u \mu_s}{\mu_u} < 1. \quad (1.1)$$

Suppose there exists a continuous Riemannian metric on \mathcal{M} such that, for all periodic points, the stable and the unstable differentials of the return map are conformal. Then g is uniformly quasi-conformal.

We note that once g in the proposition is uniformly quasi-conformal, the results of Sadovskaya in [17] imply that g is actually conformal with respect to a smooth Riemannian metric (see Corollary 3.3 below).

Let f be a diffeomorphism satisfying the assumptions of Theorem 1.1. Then it follows from the theorem that f also satisfies conditions (1.1) above, and so does any sufficiently C^1 -small perturbation of f . Thus Corollary 1.2 and Proposition 1.3 imply the following theorem.

Theorem 1.4. *Let f be a diffeomorphism as in Theorem 1.1 and let g be its C^1 -small perturbation. Suppose that there exists a continuous Riemannian metric on \mathcal{M} such that for all periodic points of g the stable and the unstable differentials of the return map are conformal. Then g is C^∞ conjugate to f .*

It would be interesting to know to what extent the assumption of Theorem 1.4 can be relaxed. Since uniform quasi-conformality is a necessary condition, the derivative of the return map at any periodic point must be diagonalizable over \mathbb{C} , and all of its stable (unstable) eigenvalues must be equal in modulus. This is equivalent to the fact that, for each periodic orbit, there exists an invariant conformal structure that, however, may vary from orbit to orbit in a non-continuous fashion.

Remark 1.5. After this paper was written, we became aware of a result by de la Llave [5, Theorem 10.3], which is similar to Proposition 1.3. In this theorem, the assumption of continuity of the Riemannian metric is essentially replaced by the assumption of its uniform boundedness. Hence the assumption of Theorem 1.4 can be similarly relaxed. The proof of Proposition 1.3 can be modified using the specification property to obtain the stronger version.

We note that if the stable and the unstable differentials of the return maps at periodic points of g are scalar multiples of identity, then they preserve *any* conformal structure, and hence the assumption of Theorem 1.4 is trivially satisfied. Thus we obtain the following corollary.

Corollary 1.6. *Let f be a diffeomorphism as in Theorem 1.1 such that, for any periodic point x ,*

$$df^m|_{E^s(x)} = a^s(x) \cdot \text{Id} \quad \text{and} \quad df^m|_{E^u(x)} = a^u(x) \cdot \text{Id}, \quad (1.2)$$

where m is the period and $a^s(x)$, $a^u(x)$ are real numbers. Let g be a C^∞ diffeomorphism of \mathcal{M} that is sufficiently C^1 -close to f , and let h be the topological conjugacy between f and g . Suppose that, for any periodic point x , the derivatives df_x^m and $dg_{h(x)}^m$ have the same Jordan normal form. Then h is a C^∞ diffeomorphism, i.e. g is C^∞ conjugate to f .

A similar result was obtained in [4] by de la Llave under the additional assumption that the subspaces $E^s(x)$ and $E^u(x)$ for all $x \in \mathcal{M}$ can be continuously identified with $\mathbb{R}^{\dim E^s(x)}$ and $\mathbb{R}^{\dim E^u(x)}$ in such a way that the restrictions of the differential of f to $E^s(x)$ and $E^u(x)$ are scalar multiples of identity. We note that this implies conformality of f with respect to a continuous metric on \mathcal{M} .

It is an open question whether the additional assumption (1.2) can be removed. This question is closely related to generalizing Proposition 1.3 and Theorem 1.4.

2. Preliminaries

In this section we briefly introduce the main notions used throughout this paper.

2.1. Anosov diffeomorphisms

Let f be a smooth diffeomorphism of a compact Riemannian manifold \mathcal{M} . The diffeomorphism f is called Anosov if there exist a decomposition of the tangent bundle $T\mathcal{M}$ into two f -invariant continuous subbundles E^s and E^u , and constants $C > 0$, $0 < \lambda < 1$, such that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \|df^n(v)\| &\leq C\lambda^n \|v\| & \text{for } v \in E^s, \\ \|df^{-n}(v)\| &\leq C\lambda^n \|v\| & \text{for } v \in E^u. \end{aligned}$$

The distributions E^s and E^u are called stable and unstable. It is well known that these distributions are tangential to the foliations W^s and W^u , respectively (see, for example, [14]). The leaves of these foliations are C^∞ injectively immersed Euclidean spaces, but in general the distributions E^s and E^u are only Hölder continuous transversally to the corresponding foliations.

2.2. Uniformly quasi-conformal diffeomorphisms

Let f be an Anosov diffeomorphism of a compact Riemannian manifold \mathcal{M} . We say that the diffeomorphism is uniformly quasi-conformal on the stable distribution or *uniformly s-quasi-conformal* if the quasi-conformal distortion

$$K^s(x, n) = \frac{\max\{\|df^n(v)\| : v \in E^s(x), \|v\| = 1\}}{\min\{\|df^n(v)\| : v \in E^s(x), \|v\| = 1\}}$$

is uniformly bounded for all $n \in \mathbb{Z}$ and $x \in \mathcal{M}$. This is equivalent to the classical definition of uniform quasi-conformality, since

$$K^s(x, n) = \limsup_{r \rightarrow 0} \frac{\sup\{d^s(f^n y, f^n x) : y \in S^s(x, r)\}}{\inf\{d^s(f^n y, f^n x) : y \in S^s(x, r)\}},$$

where d^s is the induced metric along the W^s leaves and

$$S^s(x, r) = \{y \in W^s(x) : d^s(x, y) = r\}.$$

If $K^s(x, n) = 1$ for all x and n , the diffeomorphism is called *s-conformal*. The notions of u-conformality and uniform u-quasi-conformality are defined similarly.

If a diffeomorphism is both uniformly u-quasi-conformal (u-conformal) and uniformly s-quasi-conformal (s-conformal), then it is called *uniformly quasi-conformal (conformal)*. We note that the notion of uniform quasi-conformality does not depend on the choice of a Riemannian metric on the manifold.

2.3. Conformal structures

(See [13] for more details.) A conformal structure on \mathbb{R}^n , $n \geq 2$, is a class of proportional inner products. The space \mathcal{C}^n of conformal structures on \mathbb{R}^n identifies with the space of real symmetric positive-definite $n \times n$ matrices with determinant 1, which is isomorphic to $SL(n, \mathbb{R})/SO(n, \mathbb{R})$. It is known that the space $\mathcal{C}^n = SL(n, \mathbb{R})/SO(n, \mathbb{R})$ carries a $GL(n, \mathbb{R})$ -invariant metric for which \mathcal{C}^n is a Riemannian symmetric space of non-positive curvature. Any linear isomorphism of \mathbb{R}^n induces an isometry of \mathcal{C}^n .

Now, let f be an Anosov diffeomorphism of a compact manifold \mathcal{M} . For each $x \in \mathcal{M}$, let $\mathcal{C}^s(x)$ be the space of conformal structures on $E^s(x)$. Thus we obtain a bundle \mathcal{C}^s over \mathcal{M} whose fibre over x is $\mathcal{C}^s(x)$. A continuous (smooth, measurable) section of \mathcal{C}^s is called a continuous (smooth, measurable) conformal structure on E^s . A measurable conformal structure τ on E^s is called *bounded* if the distance between $\tau(x)$ and $\tau_0(x)$ is uniformly bounded on \mathcal{M} , where τ_0 is a continuous conformal structure on E^s .

Clearly, a diffeomorphism is conformal with respect to a Riemannian metric if and only if it preserves the conformal structure associated with this metric.

2.4. Affine connections

Let \mathcal{M} be a smooth manifold. An affine connection ∇ on \mathcal{M} is a mapping that associates a vector field $\nabla_X Y$ to a pair of smooth vector fields X and Y on \mathcal{M} and satisfies the following properties,

- (1) $\nabla_{\varphi X + \psi Y} Z = \varphi \nabla_X Z + \psi \nabla_Y Z$,
- (2) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,
- (3) $\nabla_X (\varphi Y) = (X\varphi)Y + \varphi \nabla_X Y$,

where X, Y, Z are smooth vector fields and φ, ψ are smooth functions on \mathcal{M} . A connection ∇ is of class C^r , $r \geq 0$, if $\nabla_X Y$ is C^r for any two C^∞ vector fields X and Y .

3. Proof of Theorem 1.1

Let f be a C^∞ transitive uniformly quasi-conformal Anosov diffeomorphism of a compact manifold \mathcal{M} with $\dim E^u \geq 2$ and $\dim E^s \geq 2$. First we recall the following two results established in [17].

Theorem 3.1 (cf. Theorem 1.3 of [17]). *Let f be a topologically transitive uniformly u -quasi-conformal C^∞ Anosov diffeomorphism of a compact manifold \mathcal{M} . Then it is conformal with respect to a Riemannian metric on E^u , which is Hölder continuous on \mathcal{M} and C^∞ along the leaves of the unstable foliation.*

Theorem 3.2 (cf. Theorem 1.4 of [17]). *Let f be a C^∞ Anosov diffeomorphism of a compact manifold \mathcal{M} with $\dim E^u \geq 2$. Suppose it is conformal with respect to a Riemannian metric on the unstable distribution, which is continuous on \mathcal{M} and C^∞ along the leaves of the unstable foliation. Then the stable holonomy maps are conformal and the stable distribution is C^∞ .*

Since the diffeomorphism f is also uniformly s -quasi-conformal, Theorems 3.1 and 3.2 also hold for the corresponding distributions and holonomy maps. Thus both the stable and the unstable distributions are C^∞ , and both the stable and the unstable holonomy maps are C^∞ and conformal with respect to the corresponding metrics.

Let τ^u and τ^s be the conformal structures associated with the conformal metrics on E^u and E^s . Since the distribution E^u is C^∞ , and the stable holonomy maps preserve the structure τ^u and are C^∞ , we conclude that τ^u is C^∞ not only along the leaves of W^u , but also along the leaves of W^s . Thus the structure τ^u is C^∞ on the manifold \mathcal{M} . Similarly, τ^s is also C^∞ on \mathcal{M} . Normalizing these structures using C^∞ functions on \mathcal{M} , we obtain metrics on E^u and E^s that are C^∞ on \mathcal{M} . Combining these two metrics, we obtain the following corollary.

Corollary 3.3. *Let f be a transitive uniformly quasi-conformal C^∞ Anosov diffeomorphism of a compact manifold \mathcal{M} with $\dim E^u \geq 2$ and $\dim E^s \geq 2$. Then the Anosov splitting is C^∞ and f is conformal with respect to a C^∞ Riemannian metric on \mathcal{M} .*

To prove Theorem 1.1, we use a result of Benoist and Labourie, who showed in [1] that any Anosov diffeomorphism with C^∞ Anosov splitting, which preserves a C^∞ affine connection, is C^∞ conjugate to an automorphism of an infranilmanifold. The main part of our proof is a construction of an f -invariant smooth affine connection. Then Corollary 3.3 and the result of Benoist and Labourie imply that f is C^∞ conjugate to an Anosov automorphism an infranilmanifold. Since the diffeomorphism is uniformly quasi-conformal, the corresponding nilpotent group has to be abelian. Indeed, if the group is not abelian, then the Anosov automorphism must have at least two unstable Lyapunov exponents and thus can not be uniformly quasi-conformal. Since the group is abelian, the infranilmanifold is finitely covered by a torus. To complete the proof of Theorem 1.1, we will construct an f -invariant smooth affine connection.

In [8], Feres proved that any $\frac{1}{2}$ -pinched Anosov diffeomorphism preserves a unique invariant continuous affine connection. We generalize this result in Corollary 3.6. In [9], Feres noted that the exponential map of an invariant connection gives a non-stationary

local linearization. Our approach is different: we use a non-stationary local linearization to obtain an f -invariant affine connection, which is as smooth as the linearization.

Proposition 3.4. *Let f be a diffeomorphism of a compact manifold \mathcal{M} . Suppose that there exists a family of C^∞ diffeomorphisms h_x , $x \in \mathcal{M}$, from a neighbourhood U_x of x to $T_x\mathcal{M}$, such that*

- (i) $h_{fx} \circ f = df_x \circ h_x$;
- (ii) $h_x(x) = 0$ and $(dh_x)_x$ is the identity map;
- (iii) h_x depends C^r smoothly on x , $r = 0, 1, \dots, \infty$.

Then there exists a C^r f -invariant affine connection.

Proof. To obtain the f -invariant connection ∇ , we pull back the affine connection ∇^x on $T_x\mathcal{M}$ using the map h_x at each point $x \in \mathcal{M}$. More precisely, for vector fields X and Y on \mathcal{M} , we define

$$(\nabla_X Y)(x) = (h_x)_*^{-1} \nabla_{(h_x)_* X}^x (h_x)_* Y.$$

It is easy to see that ∇ is an affine f -invariant connection, which is as smooth as the dependence of h_x on x . □

Thus, to obtain a smooth f -invariant affine connection, it suffices to construct a local non-stationary linearization and show that it is smooth. First we use the following proposition from [17] to obtain linearizations in the stable and in the unstable directions separately.

Proposition 3.5 (cf. Proposition 4.1 of [17]). *Let f be a diffeomorphism of a compact Riemannian manifold \mathcal{M} and let W be a continuous invariant foliation with C^∞ leaves. Suppose that $\|df|_{TW}\| < 1$ and there exist numbers $C > 0$ and $\varepsilon > 0$ such that, for any $x \in \mathcal{M}$ and $n \in \mathbb{N}$,*

$$\|(df^n|_{T_x W})^{-1}\| \cdot \|df^n|_{T_x W}\|^2 \leq C(1 - \varepsilon)^n. \tag{3.1}$$

Then, for any $x \in \mathcal{M}$, there exists a C^∞ diffeomorphism $h_x^w : W(x) \rightarrow T_x W$ such that

- (i) $h_{fx}^w \circ f = df_x \circ h_x^w$;
- (ii) $h_x^w(x) = 0$ and $(dh_x^w)_x$ is the identity map;
- (iii) h_x^w depends continuously on x in C^∞ topology.

Clearly, condition (3.1) is satisfied for uniformly s-quasi-conformal diffeomorphisms, with W being the stable foliation. Thus we obtain a family of diffeomorphisms $h_x^s : W^s(x) \rightarrow E^s(x)$. Similarly, for a uniformly u-quasi-conformal diffeomorphism, we obtain a family $h_x^u : W^u(x) \rightarrow E^u(x)$.

If, for an Anosov diffeomorphism, there exist stable and unstable linearizations h^s and h^u as above, we can construct a local linearization $h_x : U_x \rightarrow T_x\mathcal{M}$, where U_x is a small

open neighbourhood of $x \in \mathcal{M}$ and h_x depends continuously on x . We define the map h_x as follows:

$$h_x|_{W^u(x)} = h_x^u, \quad h_x|_{W^s(x)} = h_x^s, \quad (3.2)$$

and, for $y \in W^u(x) \cap U_x$ and $z \in W^s(x) \cap U_x$, we set

$$h_x([y, z]) = h_x^u(y) + h_x^s(z), \quad (3.3)$$

where $[y, z] = W_{\text{loc}}^s(y) \cap W_{\text{loc}}^u(z)$. It is easy to see that h satisfies conditions (i)–(iii) of Proposition 3.4 with $r = 0$.

Thus, as a corollary of Propositions 3.4 and 3.5, we obtain the following statement. Since our assumption is clearly weaker than $\frac{1}{2}$ -pinching, it generalizes the result of Feres mentioned above.

Corollary 3.6. *Let f be an Anosov diffeomorphism of a compact Riemannian manifold \mathcal{M} . Suppose there exists $C > 0$ and $\varepsilon > 0$ such that, for any $x \in \mathcal{M}$,*

$$\|(df^n|_{E^s(x)})^{-1}\| \cdot \|df^n|_{E^s(x)}\|^2 \leq C(1 - \varepsilon)^n \quad \text{for } n > 0$$

and

$$\|(df^n|_{E^u(x)})^{-1}\| \cdot \|df^n|_{E^u(x)}\|^2 \leq C(1 - \varepsilon)^{|n|} \quad \text{for } n < 0.$$

Then f preserves a continuous affine connection.

So far, we have constructed a local non-stationary linearization h and an invariant continuous affine connection ∇ for the uniformly quasi-conformal Anosov diffeomorphism f . To complete the proof of Theorem 1.1, it remains to show that h_x depend smoothly on x , and hence ∇ is smooth. Since the stable and unstable foliations are C^∞ , it is clear from the definition of h_x above that it suffices to prove that the maps h_x^s and h_x^u depend smoothly on x . We will show this for h_x^s .

An important property of the maps h_x^s is that they are conformal in the following sense. Recall that τ^s is the invariant conformal structure on the stable distribution. For each $x \in \mathcal{M}$, we extend the conformal structure $\tau^s(x)$ at $0 \in E^s(x)$ to all other points of $E^s(x)$ via translations. We denote this constant (translation-invariant) conformal structure on $E^s(x)$ by σ^s . Since the conformal structure τ^s is f -invariant, σ^s is df -invariant. The following lemma from [17] shows that h_x^s is conformal, i.e. it takes τ^s on $W^s(x)$ into σ^s on $E^s(x)$. We include the proof for the sake of completeness.

Lemma 3.7 (cf. Lemma 3.1 of [17]). *If the family of maps h_x^s satisfies properties (i)–(iii) of Proposition 3.5, then h_x^s is conformal, i.e. it takes τ^s on W_x^s into σ^s on $E^s(x)$.*

Proof. For any map g and conformal structure $\rho(x)$ at a point $x \in \mathcal{M}$, we denote by $g(\rho(x))$ the push forward of $\rho(x)$ to the point $g(x)$ by dg_x . To simplify the notations, for this proof we put $h = h^s$, $\tau = \tau^s$ and $\sigma = \sigma^s$.

We need to show that, for any $y \in W^s(x)$, $h_x(\tau(y)) = \sigma(h_x(y))$. To do this, we move forward using the diffeomorphism f . First we note that, for any $\varepsilon > 0$, there exists $n > 0$ such that

$$\text{dist}(h_{f^n x}(\tau(f^n y)), \sigma(h_{f^n x}(f^n(y)))) < \varepsilon.$$

Indeed, it follows from Proposition 3.5 (iii) that the restrictions of the derivative of h_x to the ball of radius 1 around x in $W^s(x)$, $x \in \mathcal{M}$, form an equicontinuous family. Hence if $f^n y$ is sufficiently close to $f^n x$, then $dh_{f^n x}(f^n y)$ is close to $dh_{f^n x}(f^n x)$, which is the identity. Thus $h_{f^n x}(\tau(f^n y))$ is close to $h_{f^n x}(\tau(f^n x))$ and, by the definition of σ ,

$$h_{f^n x}(\tau(f^n x)) = \sigma(h_{f^n x}(f^n x)) = \sigma(h_{f^n x}(f^n y)).$$

To obtain the following equalities, we note that df^{-n} induces an isometry between the spaces of conformal structures, τ is f -invariant, σ is df -invariant and $h_x(y) = df^{-n}(h_{f^n x}(f^n y))$ by Proposition 3.5 (i). Thus

$$\begin{aligned} \varepsilon &> \text{dist}(h_{f^n x}(\tau(f^n y)), \sigma(h_{f^n x}(f^n y))) \\ &= \text{dist}(df^{-n}(h_{f^n x}(\tau(f^n y))), df^{-n}(\sigma(h_{f^n x}(f^n y)))) \\ &= \text{dist}(df^{-n}(h_{f^n x}(f^n(\tau(y)))), \sigma(df^{-n}(h_{f^n x}(f^n y)))) \\ &= \text{dist}(h_x(\tau(y)), \sigma(h_x(y))). \end{aligned}$$

As the above holds for any $\varepsilon > 0$, it follows that $h_x(\tau(y)) = \sigma(h_x(y))$. \square

Now we show that the maps h_x^s depend smoothly on the base point. We fix a point x in \mathcal{M} and consider the local coordinates given by the C^∞ map $h_x : U_x \rightarrow T_x \mathcal{M}$ defined by (3.2) and (3.3). Let us identify $T_x \mathcal{M}$ with $\mathbb{R}^n \times \mathbb{R}^m$ in such a way that $E^s(x)$ corresponds to \mathbb{R}^n , and $E^u(x)$ corresponds to \mathbb{R}^m . Then h_x identifies the neighbourhood $U_x \subset \mathcal{M}$ of x with an open neighbourhood U of 0 in $\mathbb{R}^n \times \mathbb{R}^m$. It follows from (3.3) that the local stable (unstable) manifolds correspond to subspaces parallel to \mathbb{R}^n (\mathbb{R}^m). The tangent space $T_p \mathcal{M}$, $p \in U_x$, is identified with $\mathbb{R}^n \times \mathbb{R}^m$ by $(dh_x)_p$ in such a way that 0 corresponds to $\tilde{p} = h_x(p)$. We will show that, when written in these coordinates, the maps h_p^s , $p \in U_x$, are identity maps and hence they depend smoothly on p .

Let us define the map

$$\psi_p = h_x|_{W_{\tilde{p}}^s \cap U_x} : W_{\tilde{p}}^s \cap U_x \rightarrow \mathbb{R}_{\tilde{p}}^n \cap U,$$

where $\mathbb{R}_{\tilde{p}}^n$ is the subspace through \tilde{p} parallel to \mathbb{R}^n . We denote by \tilde{h}_p^s the coordinate representation of h_p^s , i.e.

$$\tilde{h}_p^s = (d\psi_p)_p \circ h_p^s \circ \psi_p^{-1} : \mathbb{R}_{\tilde{p}}^n \cap U \rightarrow \mathbb{R}_{\tilde{p}}^n.$$

It is clear from the construction of h_x that

$$\psi_p = \tilde{H}_{0, \tilde{p}}^u \circ h_x^s \circ H_{p, x}^u,$$

where $H_{p, x}^u$ is the unstable holonomy map from $W^s(p)$ to $W^s(x)$, and $\tilde{H}_{0, \tilde{p}}^u$ is the projection from \mathbb{R}^n to $\mathbb{R}_{\tilde{p}}^n$ along \mathbb{R}^m in $\mathbb{R}^n \times \mathbb{R}^m$. The holonomy map $H_{p, x}^u$ is C^∞ and conformal by Theorem 3.2. The map h_x^s is C^∞ and conformal by Proposition 3.5 and Lemma 3.7. Hence the map ψ_p is also C^∞ and conformal. Since h_p^s is conformal, we conclude that \tilde{h}_p^s is C^∞ and conformal. We also note that

$$\tilde{h}_p^s(\tilde{p}) = \tilde{p} \quad \text{and} \quad (d\tilde{h}_p^s)_{\tilde{p}} = (d\psi_p)_p \circ (dh_p^s)_p \circ (d\psi_p^{-1})_{\tilde{p}} = \text{Id}, \quad (3.4)$$

since, by Proposition 3.5 (ii), we have $(dh_p^s)_p = \text{Id}$, where Id is the identity map.

To conclude that \tilde{h}_p^s depends smoothly on p we will use the fact that the holonomy map $H_{p,x}^u$ from $W^s(p)$ to $W^s(x)$ is defined globally on the whole stable leaf. If $\dim E^s > 2$, this fact is given by Proposition 3.8 below. If $\dim E^s = 2$, we use the assumption that \mathcal{M} is an infranilmanifold. In this case, the transitive Anosov diffeomorphism f of \mathcal{M} is known to be topologically conjugate to an Anosov *automorphism* of \mathcal{M} [10, 15], which again implies that the holonomy map $H_{p,x}^u$ from $W^s(p)$ to $W^s(x)$ is defined globally on the whole stable leaf. This is the only place in the proof where we use the extra assumption that \mathcal{M} is an infranilmanifold in the case of $\dim E^s = 2$ (or $\dim E^s = 2$). Now it is easy to see that the map ψ_p can be extended to the whole $W^s(p)$. Hence the map \tilde{h}_p^s can be extended to a conformal C^∞ map from \mathbb{R}_p^n to itself. Since $n \geq 2$, this implies that \tilde{h}_p^s is a linear map. Now it follows from equations (3.4) that \tilde{h}_p^s is the identity map. Thus we conclude that the maps h_p^s , h_p^u , and hence h_p , depend C^∞ smoothly on the base point p .

To complete the proof of Theorem 1.1, it remains to prove the following proposition.

Proposition 3.8. *Let f be a uniformly s-quasi-conformal transitive Anosov diffeomorphism of a compact manifold \mathcal{M} with dimension of the stable distribution greater than 2. Then the unstable holonomy maps are defined globally, i.e. on the whole leaves of the stable foliation.*

Proof. Recall that, by Theorems 3.1 and 3.2, the unstable holonomies are C^∞ and conformal with respect to a continuous Riemannian metric on E^s , which is smooth along the leaves of W^s . By Proposition 3.5 and Lemma 3.7, there exists a continuous (in C^∞ topology) family of C^∞ conformal maps $h_x^s : W^s(x) \rightarrow E^s(x)$ that give a non-stationary linearization of f along the stable leaves.

We note that the maps h_x^s induce a conformal affine structure on the stable leaves via identifications of $W^s(x)$ with $E^s(x)$. Indeed, for $z \in W^s(x)$, the map $h_z^s \circ (h_x^s)^{-1} : E^s(x) \rightarrow E^s(z)$ is a globally defined smooth conformal map and hence it is a conformal affine map. Thus we have a notion of a sphere in $W^s(x)$. Since the unstable holonomy maps are conformal and $\dim E^s > 2$, the holonomies map spheres to spheres. This follows from the fact that a conformal map from an open set in \mathbb{R}^n to \mathbb{R}^n , $n > 2$, is a composition of an affine map and an inversion.

The rest of the proof is an adaptation of an argument used by Ghys in [7] to study holomorphic Anosov systems. To prove the proposition, it suffices to show that, for any point $x \in \mathcal{M}$ and any nearby point $y \in W^u(x)$, the holonomy $H_{x,y}^u$ is defined on any ball in $W^s(x)$ containing x . Here, by a ball we mean a compact set in $W^s(x)$ whose boundary is a sphere. Let us fix such x and y and consider a ball B in $W^s(x)$ containing x .

We fix some Riemannian metric on \mathcal{M} and connect x and y by a shortest path $\gamma : [0, 1] \rightarrow W^u(x)$ with $x = \gamma(0)$ and $y = \gamma(1)$. Let t_0 be the supremum of $t \in [0, 1]$ such that the holonomy map $H_{x,\gamma(t)}^u$ from $W^s(x)$ to $W^s(\gamma(t))$ is defined on the whole ball B . Since B is a compact set, it is clear that the holonomy is defined from B to any sufficiently close stable leaf, and hence $t_0 > 0$. It suffices to show that $H_{x,\gamma(t_0)}^u$ is defined on the whole ball B . Indeed, in this case, $H_{x,\gamma(t_0)}^u(B)$ is compact and hence the holonomy could be extended beyond t_0 , which forces $t_0 = 1$.

If the supremum R of $\text{diam } H_{x,\gamma(t)}^u(B)$ for $0 \leq t < t_0$ is finite, then $H_{x,\gamma(t_0)}^u$ is defined on the whole ball B . This follows from the fact that, for any t sufficiently close to t_0 , the holonomy $H_{\gamma(t),\gamma(t_0)}^u$ is defined on the whole ball of radius R around $\gamma(t)$.

Suppose, to the contrary, that $\text{diam } H_{x,\gamma(t_n)}^u(B)$ tends to infinity as $t_n \rightarrow t_0$. We denote $x_n = \gamma(t_n)$. Recall that $H_{x,x_n}^u(B)$ is a ball in $W^s(x_n)$, i.e. the image of a ball $B(\tilde{z}_n)$ in $E^s(x_n)$ centred at some point \tilde{z}_n under the map $(h_{x_n}^s)^{-1}$. It is easy to see that $H_{x,x_n}^u(B)$ is the image of a ball in $E^s(z_n)$ centred at zero under the map $(h_{z_n}^s)^{-1}$, where $z_n = (h_{x_n}^s)^{-1}(\tilde{z}_n)$. This follows from the fact that $h_{z_n}^s \circ (h_{x_n}^s)^{-1}$ is a conformal affine map and thus takes $B(\tilde{z}_n)$ to a ball in $E^s(z_n)$ centred at $0 = h_{z_n}^s \circ (h_{x_n}^s)^{-1}(\tilde{z}_n)$. We note that for any point $z \in \mathcal{M}$, the images of the balls centred at zero in $E^s(z)$ under the map $(h_z^s)^{-1}$ exhaust $W^s(z)$. Hence the diameter of the largest metric ball contained in $H_{x,x_n}^u(B)$ tends to infinity.

We recall that for a transitive Anosov diffeomorphism f there exists a family $\{\mu^s\}$ of measures on the stable leaves which are conditional measures of the Bowen–Margulis measure (the unique measure of maximal entropy) [14]. These measures are invariant under the unstable holonomies, i.e. $\mu^s(V) = \mu^s(H_{x,y}^u(V))$, where V is an open subset of $W^s(x)$ with compact closure. These measures also contract uniformly under f , i.e. $\mu^s(f^n V) = e^{-hn} \mu^s(V)$, where h is the topological entropy of f . Since the measures are invariant under the unstable holonomies and the Bowen–Margulis measure is positive on open sets, it is easy to see that $\mu^s(B_1^s(x))$ is bounded away from zero, where $B_1^s(x)$ is a ball of radius 1 in $W^s(x)$. Now it follows from the uniform contraction property that $\mu^s(B_R^s(x)) \rightarrow \infty$ as $R \rightarrow \infty$ uniformly in x . This implies that $\mu^s(H_{x,x_n}^u(B)) \rightarrow \infty$, which contradicts the fact that $\mu^s(H_{x,x_n}^u(B)) = \mu^s(B)$ by holonomy invariance.

Thus we conclude that the holonomy map $H_{x,\gamma(t_0)}^u$ is defined on the whole B and hence $\gamma(t_0) = y$. Since the choice of B is arbitrary, it follows that the holonomy $H_{x,y}^u$ is defined on the whole stable leaf $W^s(x)$. □

4. A conformal Anosov automorphism of an infratorus

In this section we construct an example of a conformal Anosov automorphism of an orientable finite factor of the four-dimensional torus \mathbb{T}^4 .

We consider a group Γ of isometries of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ generated by the integral translations $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$ and an element γ such that, for $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\gamma(x, y) = (x + v, -y)$, where

$$v = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}.$$

Note that $\gamma^2 \in \mathbb{Z}^4$, and \mathbb{Z}^4 is a normal subgroup of index 2 in Γ .

It is easy to see that the group Γ acts on \mathbb{R}^4 without fixed points. Hence $\mathcal{N} = \mathbb{R}^4/\Gamma$ is a flat manifold whose double cover is \mathbb{T}^4 . Note that \mathcal{N} is orientable, since both \mathbb{Z}^4 and γ preserve the orientation of \mathbb{R}^4 . Also, \mathcal{N} is not a torus, since Γ is not abelian. Indeed, if $\beta(x, y) = (x, y + y')$, where $y' \neq (0, 0)$, then $\beta \circ \gamma \neq \gamma \circ \beta$.

Let \mathbf{A} be the direct product of an Anosov automorphism A of \mathbb{R}^2 with itself,

$$\mathbf{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : \mathbb{R}^4 \rightarrow \mathbb{R}^4, \quad \text{where } A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

To show that the action of \mathbf{A} on \mathbb{R}^4 projects to \mathcal{N} , we verify that, for any $(x, y) \in \mathbb{R}^4$, $\mathbf{A}(\Gamma(x, y)) = \Gamma(\mathbf{A}(x, y))$. Since $\det \mathbf{A} = 1$, $\mathbf{A}(\mathbb{Z}^4) = \mathbb{Z}^4$. Thus it suffices to check that $\mathbf{A}(\gamma(x, y)) \in \mathbb{Z}^4(\gamma(\mathbf{A}(x, y)))$ and hence $\mathbf{A}(\mathbb{Z}^4(\gamma(x, y))) = \mathbb{Z}^4(\gamma(\mathbf{A}(x, y)))$. This can be seen as follows:

$$\begin{aligned} \mathbf{A}(\gamma(x, y)) - \gamma(\mathbf{A}(x, y)) &= \mathbf{A}(x + v, -y) - \gamma(Ax, Ay) \\ &= (Ax + Av, -Ay) - (Ax + v, -Ay) \\ &= (Av - v, 0) \\ &= \left(\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, 0 \right) \\ &= \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \right) \in \mathbb{Z}^2 \times \mathbb{Z}^2. \end{aligned}$$

Thus we obtain a conformal Anosov automorphism of \mathcal{N} .

5. Proofs of the local rigidity results

In this section we prove our results on local rigidity. First we note that, as indicated in §1, Theorem 1.4 follows from Corollary 1.2 and Proposition 1.3, and Corollary 1.6 follows from Theorem 1.4. Below we give the proofs of Corollary 1.2 and Proposition 1.3.

5.1. Proof of Corollary 1.2

If g is Lipschitz conjugate to f , then it easily follows from the definition that g is uniformly quasi-conformal (see §2.2).

Suppose that g is uniformly quasi-conformal. Then, by Theorem 1.1, both f and g are C^∞ conjugate to affine automorphisms of a finite factor of a torus \mathbb{T}^k , where $k = \dim \mathcal{M}$. These affine automorphisms and the corresponding conjugacy lift to the torus \mathbb{T}^k . For this we note that the fundamental group of \mathcal{M} has a unique maximal abelian subgroup isomorphic to \mathbb{Z}^k . Thus it suffices to show that any two Anosov automorphisms A and B of \mathbb{T}^k that are topologically conjugate are also C^∞ conjugate. Let h be a conjugacy, i.e. a homeomorphism of \mathbb{T}^k such that $A \circ h = h \circ B$. Let H be the induced action of h on the fundamental group \mathbb{Z}^k of \mathbb{T}^k . Then H is an integral matrix with determinant ± 1 , and hence it induces an automorphism of \mathbb{T}^k . From the induced actions of A , B and h on the fundamental group \mathbb{Z}^k , we see that $A \circ H = H \circ B$. Thus H gives a smooth conjugacy between A and B . In fact, $H = h$, since the conjugacy to an Anosov automorphism is known to be unique in a given homotopy class [14].

5.2. Proof of Proposition 1.3

We will show that g is uniformly s-quasi-conformal. Uniform u-quasi-conformality of g follows in the same way.

Recall that \mathcal{C}^s is a fibre bundle over \mathcal{M} whose fibre over x is the space $\mathcal{C}^s(x)$ of conformal structures on $E^s(x)$. Let σ be the continuous conformal structure on the stable distribution induced by the Riemannian metric given in Proposition 1.3. For σ , or any other conformal structure, we denote by $g^n(\sigma(x)) \in \mathcal{C}^s(g^n x)$ the push forward of $\sigma(x) \in \mathcal{C}^s(x)$ to the point $g^n x$ by $dg^n|_{E^s(x)}$.

Since g is topologically transitive, we can consider a point x with dense orbit. Let $\tau(x) \in \mathcal{C}^s(x)$ be an arbitrary conformal structure at x . On the orbit of x we define an invariant conformal structure τ as follows: $\tau(g^n x) = g^n(\tau(x))$ for $n \in \mathbb{Z}$. We will show that the structure τ is bounded, i.e. g is uniformly s-quasi-conformal along the orbit of x . Since this orbit is dense, this easily implies that g is uniformly s-quasi-conformal on \mathcal{M} .

We denote by $l(x)$ the distance between the conformal structures $\tau(x)$ and $\sigma(x)$. We will show that the function l is uniformly continuous on the orbit of x , and hence extends to a continuous function on \mathcal{M} . This implies that l is bounded. Let $y = g^m x$ and suppose that $g^n y$ is close enough to y to apply the Anosov closing lemma (see [14, Theorem 6.4.15]). Then there exists a periodic point $z \in \mathcal{M}$ with $g^n z = z$ such that

$$\text{dist}(g^i y, g^i z) \leq k \cdot \text{dist}(y, g^n y) \quad \text{for } i = 0, 1, \dots, n,$$

where k is a uniform constant. Then it follows from Lemma 5.1 below that

$$\|(dg_z^n)^{-1} \circ dg_y^n - \text{Id}\| \leq Ck \cdot \text{dist}(y, g^n y). \tag{5.1}$$

Since the differential $dg^n|_{E^s(y)}$ induces an isometry on the space of conformal structures and τ is invariant, we obtain

$$\begin{aligned} l(g^n y) &= \text{dist}(\tau(g^n y), \sigma(g^n y)) \\ &= \text{dist}(g^n(\tau(y)), \sigma(g^n y)) \\ &\leq \text{dist}(g^n(\tau(y)), g^n(\sigma(y))) + \text{dist}(g^n(\sigma(y)), \sigma(g^n y)) \\ &= \text{dist}(\tau(y), \sigma(y)) + \text{dist}(g^n(\sigma(y)), \sigma(g^n y)) \\ &= l(y) + \text{dist}(g^n(\sigma(y)), \sigma(g^n y)). \end{aligned}$$

To estimate the last term we note that $g^n(\sigma(z)) = \sigma(g^n(z)) = \sigma(z)$. We also recall that σ is continuous on \mathcal{M} and hence it is bounded and uniformly continuous. Let $\omega(\varepsilon)$ be its modulus of continuity, i.e. if $\text{dist}(x, y) < \varepsilon$, then $\text{dist}(\sigma(x), \sigma(y)) < \omega(\varepsilon)$, and $\omega(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then, using (5.1), we obtain

$$\begin{aligned} \text{dist}(g^n(\sigma(y)), \sigma(g^n y)) &\leq \text{dist}(g^n(\sigma(y)), g^n(\sigma(z))) + \text{dist}(\sigma(g^n z), \sigma(g^n y)) \\ &\leq \text{dist}(((dg_z^n)^{-1} \circ dg_y^n)_* \sigma(y), \sigma(z)) + \omega(\text{dist}(z, g^n y)) \\ &\leq \text{dist}(((dg_z^n)^{-1} \circ dg_y^n)_* \sigma(y), \sigma(y)) + \text{dist}(\sigma(y), \sigma(z)) + \omega(k \cdot \text{dist}(y, g^n y)) \\ &\leq k_1 k C \cdot \text{dist}(y, g^n y) + 2\omega(k \cdot \text{dist}(y, g^n y)), \end{aligned}$$

where the constant k_1 depends on the bounded structure σ . Thus we see that

$$|l(y) - l(g^n y)| \leq k_1 k C \cdot \text{dist}(y, g^n y) + 2\omega(k \cdot \text{dist}(y, g^n y)) =: \tilde{\omega}(\text{dist}(y, g^n y)).$$

Clearly, $\tilde{\omega}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and hence the function l is uniformly continuous. This implies that l is bounded and thus the conformal structure τ is bounded along the dense orbit of x , i.e. the quasi-conformal distortion $K^s(x, n) \leq K$ for all $n \in \mathbb{Z}$ (see § 2.2). Since the orbit of x is dense and $K^s(y, n)$ depends continuously on y for any fixed n , it is easy to see that $K^s(y, n) \leq 2K$ for any $y \in \mathcal{M}$ and $n \in \mathbb{Z}$. Thus g is uniformly s -quasi-conformal.

To complete the proof of the proposition, it remains to prove the following lemma.

Lemma 5.1. *Let g be an Anosov diffeomorphism of a compact manifold \mathcal{M} satisfying condition (1.1) of Proposition 1.3. Then there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$, $x, y \in \mathcal{M}$ and $n \in \mathbb{N}$ with*

$$\text{dist}(g^i(x), g^i(y)) < \varepsilon \quad \text{for } 0 \leq i \leq n,$$

we have $\|(dg_x^n)^{-1} \circ dg_y^n - \text{Id}\| \leq C\varepsilon$.

Here, to consider the composition of the derivatives, we identify the tangent spaces at nearby points preserving the Anosov splitting. Since condition (1.1) implies, in particular, that the Anosov splitting is C^1 (see [11, 12]), this identification can be also chosen C^1 .

Proof. Since the differential dg_x is the direct sum of the stable differential $dg|_{E^s(x)}$ and the unstable differential $dg|_{E^u(x)}$, it suffices to prove the lemma for these restrictions. We will prove the lemma for the stable differential, and to simplify the notations we will write dg_x for $dg|_{E^s(x)}$.

If ε_0 is small enough, there exists a unique point $z \in W_{\text{loc}}^u(x) \cap W_{\text{loc}}^s(y)$ with

$$\text{dist}(g^i(x), g^i(z)) < K\varepsilon \quad \text{and} \quad \text{dist}(g^i(z), g^i(y)) < K\varepsilon \quad \text{for } 0 \leq i \leq n.$$

Thus it is sufficient to prove the lemma for x and y lying on the same stable or on the same unstable manifold. We use the notations $x^i = g^i(x)$ and $y^i = g^i(y)$ for $i = 0, 1, \dots, n$.

First we consider the case when $y \in W^s(x)$. Then

$$\begin{aligned} (dg_x^n)^{-1} \circ dg_y^n &= (dg_x^{n-1})^{-1} \circ ((dg_{x^{n-1}})^{-1} \circ dg_{y^{n-1}}) \circ dg_y^{n-1} \\ &= (dg_x^{n-1})^{-1} \circ (\text{Id} + r_{n-1}) \circ dg_y^{n-1} \\ &= (dg_x^{n-1})^{-1} \circ dg_y^{n-1} + (dg_x^{n-1})^{-1} \circ r_{n-1} \circ dg_y^{n-1} \\ &= \dots = \text{Id} + \sum_{i=0}^{n-1} (dg_x^i)^{-1} \circ r_i \circ dg_y^i, \end{aligned}$$

where we write $(dg_{x^i})^{-1} \circ dg_{y^i} = \text{Id} + r_i$. Since the stable differential is Lipschitz continuous and $y \in W^s(x)$, we have

$$\|r_i\| \leq L \cdot \text{dist}(x^i, y^i) \leq L \cdot \varepsilon \cdot \mu_s^i.$$

Now, using the first equation in (1.1), we conclude that

$$\begin{aligned} \|\text{Id} - (dg_x^n)^{-1} \circ dg_y^n\| &\leq \sum_{i=0}^{n-1} \|(dg_x^i)^{-1}\| \cdot \|r_i\| \cdot \|dg_y^i\| \\ &\leq \sum_{i=0}^{n-1} \nu_s^{-i} \cdot L\varepsilon \mu_s^i \cdot \mu_s^i \\ &\leq L\varepsilon \sum_{i=0}^{n-1} \left(\frac{\mu_s^2}{\nu_s}\right)^i \\ &\leq C\varepsilon. \end{aligned}$$

Similarly, we consider the case when $y \in W^u(x)$,

$$\begin{aligned} dg_x^n \circ (dg_y^n)^{-1} &= dg_{x^1}^{n-1} \circ (dg_x \circ (dg_y)^{-1}) \circ (dg_{y^1}^{n-1})^{-1} \\ &= dg_{x^1}^{n-1} \circ (\text{Id} + r_{n-1}) \circ (dg_{y^1}^{n-1})^{-1} \\ &= dg_{x^1}^{n-1} \circ (dg_{y^1}^{n-1})^{-1} + dg_{x^1}^{n-1} \circ r_{n-1} \circ (dg_{y^1}^{n-1})^{-1} \\ &= \dots = \text{Id} + \sum_{i=0}^{n-1} dg_{x^{n-i}}^i \circ r_i \circ (dg_{y^{n-i}}^i)^{-1}, \end{aligned}$$

where we write $dg_{x^{n-i}} \circ (dg_{y^{n-i}})^{-1} = \text{Id} + r_i$. Since the stable differential is Lipschitz continuous and $y \in W^u(x)$, we have

$$\|r_i\| \leq L \cdot \text{dist}(x^{n-i}, y^{n-i}) \leq L \cdot \mu_u^{-i} \cdot \text{dist}(x^n, y^n) \leq L \cdot \varepsilon \cdot \mu_u^{-i}.$$

Now, using the second equation in (1.1), we conclude that

$$\begin{aligned} \|\text{Id} - dg_x^n \circ (dg_y^n)^{-1}\| &\leq \sum_{i=0}^{n-1} \|dg_{x^{n-i}}^i\| \cdot \|r_i\| \cdot \|(dg_{y^{n-i}}^i)^{-1}\| \\ &\leq \sum_{i=0}^{n-1} \mu_s^i \cdot L\varepsilon \mu_u^{-i} \cdot \nu_s^{-i} \\ &\leq L\varepsilon \sum_{i=0}^{n-1} \left(\frac{\mu_s}{\mu_u \nu_s}\right)^i \leq C\varepsilon. \end{aligned}$$

This completes the proof of the lemma. □

Acknowledgements. We thank Ralf Spatzier for helpful discussions. B.K. was partially supported by NSF grant DMS 0140513.

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