

## On pointwise dimension of non-hyperbolic measures

BORIS KALININ and VICTORIA SADOVSKAYA

*The University of Michigan, Ann Arbor, MI 48109, USA*  
(e-mail: {kalinin,sadovska}@umich.edu)

(Received 20 February 2001 and accepted in revised form 12 May 2001)

*Abstract.* We construct a diffeomorphism preserving a non-hyperbolic measure whose pointwise dimension does not exist almost everywhere. In the one-dimensional case we also show that such diffeomorphisms are typical in certain situations.

### 1. Introduction

We consider an ergodic measure  $\mu$  invariant under a diffeomorphism  $f$  of a compact Riemannian manifold  $\mathcal{M}$ . Such a measure  $\mu$  is called *hyperbolic* if all its Lyapunov exponents are different from zero. The main goal of this paper is to show that hyperbolicity of a measure is essential for existence of its pointwise dimension.

We recall that the *pointwise dimension* at a point  $x$  of a Borel measure  $\mu$  on a metric space  $X$  is defined as the following limit:

$$d_\mu(x) = \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r},$$

where  $B(x, r)$  is a ball of radius  $r$  centered at  $x \in X$ . This limit does not exist in general. However the upper and lower pointwise dimensions  $\bar{d}_\mu(x)$  and  $\underline{d}_\mu(x)$  can be defined at any point  $x$  as corresponding upper and lower limits.

The study of pointwise dimension of hyperbolic measures in [3] has led to the problem known as the Eckmann–Ruelle conjecture. The complete affirmative solution of this problem was obtained by Barreira *et al* [2] as follows.

**THEOREM.** *Let  $f : \mathcal{M} \rightarrow \mathcal{M}$  be a  $C^{1+\alpha}$  diffeomorphism of a compact smooth Riemannian manifold  $\mathcal{M}$ . If  $\mu$  is a hyperbolic ergodic measure for  $f$  then the pointwise dimension of  $\mu$  exists for  $\mu$ -almost every  $x \in \mathcal{M}$  and is constant.*

One may ask what happens if the requirement that  $\mu$  is hyperbolic is omitted. The first result along this direction was obtained by Ledrappier and Misiurewicz in [7]. They constructed an example of a  $C^r$ -smooth map of an interval preserving an ergodic measure with zero Lyapunov exponent whose pointwise dimension does not exist almost everywhere. For a discussion of the above results see [8].

In this paper, we consider circle diffeomorphisms with irrational rotation number which are known to be uniquely ergodic and have zero Lyapunov exponent.

In §2 we prove the genericity of circle diffeomorphisms  $f$  with irrational rotation number whose unique invariant measure  $\mu_f$  has lower pointwise dimension 0 and upper pointwise dimension 1 for  $\mu_f$ -almost every point in  $S^1$ . We also prove density of circle diffeomorphisms with irrational rotation number and given lower pointwise dimension of the unique invariant measure.

In §3 we show that circle homeomorphisms  $g$  with given upper and lower pointwise dimension of the unique invariant measure  $\mu_g$  are dense in the set of all circle homeomorphisms with any given irrational rotation number.

In §4 we prove the genericity of analytic circle diffeomorphisms  $f$  with irrational rotation number whose unique invariant measure  $\mu_f$  has lower pointwise dimension 0 and upper pointwise dimension 1 for  $\mu_f$ -almost every point.

Let  $f$  be a circle diffeomorphism such that its unique invariant measure  $\mu_f$  has lower pointwise dimension 0 and upper pointwise dimension 1 for  $\mu_f$ -almost every point. Consider the direct product of a volume-preserving Anosov diffeomorphism and  $f$ . It is easy to see that we obtain a partially hyperbolic diffeomorphism with only one zero Lyapunov exponent. The product measure is ergodic with respect to this diffeomorphism and its pointwise dimension does not exist almost everywhere. This shows that hyperbolicity of the measure is crucial in the Eckmann–Ruelle conjecture.

## 2. Circle diffeomorphisms

We adopt the following notation. Denote by  $D_I^r \subset \text{Diff}^r(S^1)$  the set of all  $C^r$  circle diffeomorphisms with irrational rotation number (see [6] for a definition and properties of rotation number).

Let  $Y^r \subset D_I^r$  be the set of all  $C^r$  circle diffeomorphisms  $f$  with irrational rotation number satisfying the following properties:

- (1)  $\underline{d}_\mu(x) = 0$  and  $\overline{d}_\mu(x) = 1$  for  $\mu$ -a.e.  $x \in S^1$ ;
- (2)  $\dim_H \mu = \underline{\dim}_B \mu = 0$  and  $\overline{\dim}_B \mu = 1$ ;

where  $\mu$  is the invariant measure for  $f$ .

We recall the following definitions of Hausdorff, upper and lower box dimensions of a Borel probability measure  $\mu$ :

$$\begin{aligned} \dim_H \mu &= \inf\{\dim_H X : \mu(X) = 1\}, \\ \underline{\dim}_B \mu &= \liminf_{\varepsilon \rightarrow 0} \{\underline{\dim}_B X : \mu(X) > 1 - \varepsilon\}, \\ \overline{\dim}_B \mu &= \liminf_{\varepsilon \rightarrow 0} \{\overline{\dim}_B X : \mu(X) > 1 - \varepsilon\} \end{aligned}$$

(see [4] for a definition and properties of Hausdorff and box dimensions).

Our main results for circle diffeomorphisms are Theorem 2.1, Corollary 2.1 and Theorem 2.2.

**THEOREM 2.1.** *For any  $0 \leq r \leq \infty$ ,  $Y^r$  is a residual subset of both  $D_1^r$  and  $\overline{D_1^r}$  (the closure of  $D_1^r$  in  $C^r$ -topology).*

Let  $D_\tau^r$  be the set of all  $C^r$  circle diffeomorphisms with rotation number  $\tau$ .

**COROLLARY 2.1.** *For any  $0 \leq r \leq \infty$ , there exists a set  $T^r \subset [0, 1] \setminus \mathbb{Q}$  which is a residual subset of  $[0, 1]$  such that for any  $\tau \in T^r$ ,  $Y^r \cap D_\tau^r$  is a residual subset of  $D_\tau^r$ .*

*Remark 2.1.* Recall that a number  $\tau$  is called Diophantine if it satisfies the following condition. There exist  $\delta > 0$  and  $K > 0$  such that for any  $p/q \in \mathbb{Q}$ ,

$$|\tau - p/q| > \frac{K}{|q|^{2+\delta}}.$$

Let  $f$  be a  $C^{2+\varepsilon}$  circle diffeomorphism, where  $\varepsilon > 0$ , and its rotation number  $\tau$  satisfies the Diophantine condition with some  $K > 0$  and  $0 < \delta < \varepsilon$ . Then  $f$  is conjugate to the rotation by  $\tau$  via a  $C^1$  diffeomorphism (see [5]). This implies that the pointwise dimension of the invariant measure for  $f$  exists at every point  $x \in S^1$  and is equal to 1.

Note that for any  $\delta > 0$  the set of all numbers satisfying the Diophantine condition with some  $K > 0$  has full Lebesgue measure. Therefore, the set  $T^r$  has zero Lebesgue measure for any  $r > 2$ . We can also show that  $\dim_H T^r \leq 2/r$  for any  $2 < r < \infty$ , and  $\dim_H T^\infty = 0$ .

The following theorem shows that any given number  $\beta, 0 < \beta < 1$ , can be the value of the lower pointwise dimension of the invariant measure for a circle diffeomorphism.

**THEOREM 2.2.** *For any given  $0 < \beta < 1$  and  $0 \leq r \leq \infty$  the set of all  $C^r$  circle diffeomorphisms  $f$  with irrational rotation number satisfying the following properties:*

- (1)  $\underline{d}_\mu(x) = \beta$  and  $\overline{d}_\mu(x) = 1$  for  $\mu$ -a.e.  $x \in S^1$ ;
- (2)  $\dim_H \mu = \underline{\dim}_B \mu = \beta$  and  $\overline{\dim}_B \mu = 1$ ;

*is a dense subset of  $D_1^r$ .*

Note that the set of diffeomorphisms described in Theorem 2.2 is not residual.

We begin with a construction of a uniquely ergodic circle diffeomorphism which is close to a given diffeomorphism and whose invariant measure  $\mu$  does not have pointwise dimension almost everywhere. Our construction is closely related to the construction in [6] of circle diffeomorphisms conjugated to rotations via maps with specific degrees of regularity. The latter construction is based on a method developed by Anosov and Katok in [1] to construct examples of diffeomorphisms with specific ergodic properties.

**PROPOSITION 2.1.** *Let  $f_* : S^1 \rightarrow S^1$  be a  $C^\infty$  circle diffeomorphism such that  $f_* = h_*^{-1} \circ R_{\tau_*} \circ h_*$ , where  $h_*$  is a  $C^\infty$  circle diffeomorphism and  $R_{\tau_*}$  is a circle rotation by  $\tau_*$ .*

*Then in any  $C^\infty$  neighborhood of  $f_*$  there exists a  $C^\infty$  diffeomorphism  $f : S^1 \rightarrow S^1$  with irrational rotation number such that for its unique invariant measure  $\mu$ ,  $\underline{d}_\mu(x) = 0$  and  $\overline{d}_\mu(x) = 1$  for  $\mu$ -a.e.  $x \in S^1$ .*

*Proof.* The desired diffeomorphism  $f$  will be obtained as a limit of a sequence of diffeomorphisms  $f_n = h_n^{-1} \circ R_{\tau_n} \circ h_n$ , where  $\tau_n = p_n/q_n$  is a rational number and  $h_n$  is a  $C^\infty$  diffeomorphism of  $S^1$ .

The sequences  $\tau_n$  and  $h_n$  will be defined inductively as follows. We take  $h_0 = h_*$  and a rational number  $\tau_0$  close to  $\tau_*$ . Once  $\tau_{n-1} = p_{n-1}/q_{n-1}$  and  $h_{n-1}$  are chosen, we construct  $h_n$  as the composition  $h_n = A_n \circ h_{n-1}$ . The diffeomorphism  $A_n$  will be constructed in the form  $A_n = \text{Id} + a_n$ , where  $\text{Id}$  is the identity map and  $a_n$  is a  $1/q_{n-1}$ -periodic  $C^\infty$  function on  $S^1$  such that  $a_n$  is zero in disjoint neighborhoods of points  $k/q_{n-1}$ ,  $k = 1, \dots, q_{n-1}$ . A particular choice of  $A_n$  will be described later. Once  $A_n$  is constructed we choose  $\tau_n$  in the form  $\tau_n = \tau_{n-1} + (1/K_n q_{n-1})$ , where  $K_n$  is an integral number. We choose  $K_n$  large enough as follows to ensure  $C^\infty$  convergence of diffeomorphisms  $f_n$  and  $C^0$  convergence of diffeomorphisms  $h_n$ .

The  $C^0$  distance between  $h_n$  and  $h_{n-1}$  (and between  $h_n^{-1}$  and  $h_{n-1}^{-1}$ ) is bounded by  $1/q_{n-1}$ , the period of  $a_n$ . Therefore the sequence of diffeomorphisms  $h_n$  converges in  $C^0$  topology to a homeomorphism  $h = \lim_{n \rightarrow \infty} h_n$  if the sequence  $q_n$  grows sufficiently fast. This can be easily ensured by choosing  $K_n$  large enough.

Since  $R_{p_{n-1}/q_{n-1}}$  and  $A_n^{-1}$  commute due to the form in which  $A_n$  is constructed we can rewrite  $f_n$  in the following way:

$$\begin{aligned} f_n &= h_n^{-1} \circ R_{\tau_n} \circ h_n = h_{n-1}^{-1} \circ A_n^{-1} \circ R_{p_{n-1}/q_{n-1}} \circ R_{1/K_n q_{n-1}} \circ A_n \circ h_{n-1} \\ &= h_{n-1}^{-1} \circ R_{p_{n-1}/q_{n-1}} \circ A_n^{-1} \circ R_{1/K_n q_{n-1}} \circ A_n \circ h_{n-1}. \end{aligned}$$

So we see that given a map  $A_n$  in the described form we can choose  $K_n$  so large that the map  $A_n^{-1} \circ R_{1/K_n q_{n-1}} \circ A_n$  is close to  $\text{Id}$  in  $C^\infty$ . It follows that we can make  $f_n$  as close to  $f_{n-1}$  in  $C^\infty$  as we wish. This allows us to choose any  $A_n$  within described restrictions and then choose  $K_n$  so that the sequence  $f_n$  converges in  $C^\infty$  and its limit  $f$  is as close to  $f_0$  as we wish. Taking  $\tau_0$  close to  $\tau_*$  we can make  $f$  close to  $f_*$ .

Note that for the diffeomorphism  $f$  the rotation number  $\tau = \lim_{n \rightarrow \infty} \tau_n$  is irrational once  $K_n$  grow to infinity. Indeed, suppose that  $\tau = p/q \in \mathbb{Q}$ . Then

$$\tau - \tau_n = \frac{p}{q} - \frac{p_n}{q_n} = \frac{pq_n - qp_n}{qq_n} \geq \frac{1}{qq_n}.$$

On the other hand,

$$\tau - \tau_n = \sum_{i=1}^{\infty} \frac{1}{q_n K_{n+1} \cdots K_{n+i}} \leq \frac{1}{q_n} \sum_{i=1}^{\infty} \frac{1}{K_{n+1}^i} = \frac{1}{q_n(K_{n+1} - 1)},$$

which contradicts the previous estimate if  $n$  is sufficiently large.

We now specify the choice of  $A_n$ . Let  $\mu$  be the invariant measure for  $f$ . We note that  $h$  is the distribution function of  $\mu$ , i.e.  $\mu([x_1, x_2]) = h(x_2) - h(x_1)$  for any interval  $[x_1, x_2] \subset S^1$ . Let  $\Delta h(x, r) = h(x+r) - h(x-r)$ . Then

$$\bar{d}_\mu(x) = \limsup_{r \rightarrow 0} \frac{\log \Delta h(x, r)}{\log r} \quad \text{and} \quad \underline{d}_\mu(x) = \liminf_{r \rightarrow 0} \frac{\log \Delta h(x, r)}{\log r}.$$

We think of  $A_n$  and  $a_n$  as  $C^\infty$  functions on the unit interval. Recall that  $a_n$  is periodic with period  $s_n = 1/q_{n-1}$  and  $A_n$  is monotone. We would like to concentrate most of the

growth of  $A_n$  on a set  $\tilde{E}_n = \bigcup_{i=1}^{q_{n-1}} I_n^i$ , where  $I_n^i$  is a subinterval of  $((i-1)/q_{n-1}, i/q_{n-1})$  of length  $d_n$ . More precisely, we choose  $A_n$  such that on each  $I_n^i$  it is linear with the slope  $d_n^{-1}s_n(1-2^{-n})$ .

Let  $E_n$  be the preimage of  $\tilde{E}_n$  under  $h_{n-1}$ . Then  $h_n(E_n) = A_n(\tilde{E}_n)$  and hence

$$\mu_n(E_n) > 1 - 2^{-n} \tag{1}$$

where  $\mu_n$  is the measure with the distribution function  $h_n$ .

Now we will show how to choose a length  $d_n$  and two ‘scales’  $r_n$  and  $\tilde{r}_n$ ,  $n \geq 0$ , such that

$$\frac{\log \Delta h_n(x, r_n)}{\log r_n} < \frac{1}{n} \quad \text{for any } x \in E_n, \tag{2}$$

$$\frac{\log \Delta h_n(x, \tilde{r}_n)}{\log \tilde{r}_n} > 1 - \frac{1}{n} \quad \text{for any } x \in [0, 1]. \tag{3}$$

This means that for the measure  $\mu_n$  the pointwise dimension ‘on the scale  $r_n$ ’ is less than  $n^{-1}$  on a set of  $\mu_n$ -measure at least  $1 - 2^{-n}$  and ‘on the scale  $\tilde{r}_n$ ’ it is at least  $1 - n^{-1}$ .

Let us introduce the following notations:

$$m_{n-1} = \min_{[0,1]} h'_{n-1} \quad \text{and} \quad M_{n-1} = \max_{[0,1]} h'_{n-1}.$$

Note that the set  $E_n$  consists of  $q_{n-1}$  intervals whose lengths are bounded above by  $d_n/m_{n-1}$ . It follows that for  $r_n = d_n/m_{n-1}$  and any  $x \in E_n$ ,

$$\frac{\log \Delta h_n(x, r_n)}{\log r_n} \leq \frac{\log(s_n(1-2^{-n}))}{\log d_n - \log m_{n-1}} \xrightarrow{d_n \rightarrow 0} 0,$$

so we can take  $d_n$  so small that (2) holds. This completes the description of the choice of  $d_n$  and  $r_n$  and the construction of  $A_n$ .

Note that  $\Delta h_n(x, r) \leq 2rM_n$ . This implies that

$$\frac{\log \Delta h_n(x, r)}{\log r} \geq 1 + \frac{\log(2M_n)}{\log r}.$$

Since  $\log(2M_n)/\log r \rightarrow 0$  as  $r \rightarrow 0$ , there exists  $\tilde{r}_n$  satisfying (3).

The distance between  $h_n$  and  $h$  is bounded by  $\sum_{i=n}^{\infty} 1/q_i$ . Since  $K_i$ ,  $i \geq n$ , can be taken as large as we wish we may assume that  $h$  is so close to  $h_n$  in  $C^0$  topology that the following properties hold true for the limit function  $h$ :

$$\mu(E_n) > 1 - 2^{-n+1}, \tag{1'}$$

$$\frac{\log \Delta h(x, r_n)}{\log r_n} < \frac{2}{n} \quad \text{for any } x \in E_n, \tag{2'}$$

$$\frac{\log \Delta h(x, \tilde{r}_n)}{\log \tilde{r}_n} > 1 - \frac{2}{n} \quad \text{for any } x \in [0, 1]. \tag{3'}$$

Thus for any  $n > 0$ , there exists a set  $E_n$  and two ‘scales’  $r_n$  and  $\tilde{r}_n$  satisfying (1')–(3'). Obviously  $r_n, \tilde{r}_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Take  $x \in [0, 1]$ . If for any  $N > 0$  there exists  $n > N$  such that  $x \in E_n$ , then  $\underline{d}_\mu(x) = 0$  and  $\overline{d}_\mu(x) \geq 1$ .

Otherwise,  $x \in J = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} ([0, 1] \setminus E_n)$ . However  $\mu(\bigcap_{n=m}^{\infty} ([0, 1] \setminus E_n)) = 0$  by (2'). Hence  $\mu(J) = 0$  and we conclude that for  $\mu$ -almost all  $x \in S^1$ ,  $\underline{d}_\mu(x) = 0$  and  $\overline{d}_\mu(x) \geq 1$ .

It remains to note that  $\overline{d}_\mu(x) \leq 1$ ,  $\mu$ -almost everywhere. This fact is probably well known and we include the following lemma for the sake of completeness.

LEMMA 2.1. *Let  $\mu$  be a Borel probability measure on  $S^1$ . Then  $\overline{d}_\mu(x) \leq 1$  for  $\mu$ -a.e.  $x \in S^1$ .*

*Proof.* The function  $\overline{d}_\mu(x)$  is measurable. If  $\overline{d}_\mu(x) > 1$  on a set of positive measure, then there exists  $\delta > 0$  and a set  $X \subset S^1$  of positive measure such that  $\overline{d}_\mu(x) \geq 1 + 2\delta$  for all  $x \in X$ . It follows from the definition of the upper pointwise dimension that for any  $\varepsilon > 0$  and any  $x \in X$  there exists  $r(x) \leq \varepsilon$  such that  $\mu(B(x, r(x))) \leq r(x)^{1+\delta}$ , where  $B(x, r(x))$  is the interval in  $S^1$  centered at  $x$  of length  $2r(x)$ . Since  $X \subset \bigcup_{x \in X} B(x, \frac{1}{4}r(x)) \subset S^1$ , by the Vitali Covering Lemma there exists an at most countable subset  $\{x_n\}_{n \geq 1}$  of  $X$  such that  $X \subset \bigcup_n B(x_n, r(x_n))$  and the balls  $B(x_n, \frac{1}{4}r(x_n))$  are disjoint. Then

$$\mu(X) \leq \sum_n \mu(B(x_n, r(x_n))) \leq \sum_n r(x_n)^{1+\delta} \leq \varepsilon^\delta \sum_n r(x_n)$$

and hence

$$\sum_n \frac{1}{4}r(x_n) \geq \frac{\mu(X)}{4\varepsilon^\delta} > 1$$

for  $\varepsilon$  sufficiently small. This contradicts the fact that  $B(x_n, \frac{1}{4}r(x_n))$  are disjoint intervals in  $S^1$ . □

It follows that  $\overline{d}_\mu(x) = 1$  for  $\mu$ -a.e.  $x \in S^1$ . This completes the proof of Proposition 2.1. □

We now construct a  $C^\infty$  circle diffeomorphism  $f$  with irrational rotation number such that for its unique invariant measure  $\mu$ , the lower pointwise dimension is equal to a given number  $\beta$ ,  $0 < \beta < 1$ , and the upper pointwise dimension is equal to 1  $\mu$ -almost everywhere.

PROPOSITION 2.2. *Let  $f_* : S^1 \rightarrow S^1$  be a  $C^\infty$  circle diffeomorphism such that  $f_* = h_*^{-1} \circ R_{\tau_*} \circ h_*$ , where  $h_*$  is a  $C^\infty$  circle diffeomorphism and  $R_{\tau_*}$  is a circle rotation by  $\tau_*$ .*

*Given  $\beta$ ,  $0 < \beta < 1$ , in any  $C^\infty$  neighborhood of  $f_*$  there exists a  $C^\infty$  diffeomorphism  $f : S^1 \rightarrow S^1$  with irrational rotation number  $\tau$  such that:*

- (1)  *$f$  is conjugate to the rotation  $R_\tau$ ;*
- (2) *the conjugacy map  $h$  is Hölder continuous with Hölder exponent  $\beta$ ;*
- (3) *if  $\mu$  is the invariant measure for  $f$  then  $\underline{d}_\mu(x) = \beta$  and  $\overline{d}_\mu(x) = 1$  for  $\mu$ -a.e.  $x \in S^1$ .*

*Proof.* We follow the same approach as in the proof of Proposition 2.1, but we would like to make  $\log \Delta h_n(x, r_n) / \log r_n$  close to  $\beta$  rather than to 0. For this we make the following modifications.

We choose the period  $s_n$  of the function  $a_n$  smaller than  $1/q_{n-1}$  in the form  $s_n = 1/l_n q_{n-1}$ . Then we take  $\tilde{E}_n = \bigcup_{i=1}^{l_n q_n} I_n^i$ , where  $I_n^i$  is a subinterval of  $((i-1)/l_n q_{n-1}, i/l_n q_{n-1})$  of length  $d_n$ . We again concentrate most of the growth of  $A_n$  on  $\tilde{E}_n$ . We take

$A_n$  to be linear on each interval  $I_n^i$  with the slope  $d_n^{-1}s_n(1 - 2^{-n})$ . We may also assume that  $s_n/d_n$  is the upper bound for the derivative of  $A_n$ . Let us again introduce the following notations:

$$m_{n-1} = \min_{[0,1]} h'_{n-1} \quad \text{and} \quad M_{n-1} = \max_{[0,1]} h'_{n-1}.$$

The preimage  $E_n$  of  $\tilde{E}_n$  under  $h_{n-1}$  consists of  $l_n q_{n-1}$  intervals whose lengths are bounded above by  $d_n/m_{n-1}$  and below by  $d_n/M_{n-1}$ . Then for any  $x \in E_n$  and  $r_n = d_n/m_{n-1}$  we have

$$\Delta h_n(x, r_n) \geq s_n(1 - 2^{-n}) \quad \text{and} \quad \Delta h_n(x, r_n) \leq \frac{s_n}{d_n} M_{n-1} 2r_n = 2s_n \frac{M_{n-1}}{m_{n-1}},$$

where  $\Delta h_n(x, r_n) = h_n(x + r_n) - h_n(x - r_n)$ . Hence

$$\frac{\log s_n}{\log r_n} + \frac{\log(2M_{n-1}/m_{n-1})}{\log r_n} \leq \frac{\log \Delta h_n(x, r_n)}{\log r_n} \leq \frac{\log s_n}{\log r_n} + \frac{\log(1 - 2^{-n})}{\log r_n}.$$

We note that the error terms

$$\frac{\log(2M_{n-1}/m_{n-1})}{\log r_n} \quad \text{and} \quad \frac{\log(1 - 2^{-n})}{\log r_n}$$

are small once  $d_n$  is chosen so small that  $r_n = d_n/m_{n-1}$  is small enough. Now we can choose  $d_n$  small and  $l_n$  large to satisfy the following properties:

- (1) the absolute values of the error terms are less than  $n^{-1}$ ;
- (2)  $(\log s_n)/(\log r_n) = \beta + n^{-1}$ ;
- (3)  $r_n \leq (M_{n-1} + 1)^{-n}$ ,  $r_n \leq (m_{n-1}/M_{n-1})^n$ ,  $s_n \leq 2^{-(n+1)}s_{n-1}$ ,

where  $s_n = 1/l_n q_{n-1}$ .

The third property will be used to prove Hölder continuity of  $h$ .

So we conclude that for any  $x \in E_n$ :

$$\beta < \frac{\log \Delta h_n(x, r_n)}{\log r_n} < \beta + \frac{2}{n}.$$

This means that for the measure  $\mu_n$  the pointwise dimension ‘on the scale  $r_n$ ’ is about  $\beta$  on a set of large  $\mu_n$ -measure.

Now it follows in the same way as in the previous proposition that  $\underline{d}_\mu(x) \leq \beta$  and  $\overline{d}_\mu(x) = 1$  for  $\mu$ -a.a.  $x \in S^1$ , where  $\mu$  is the unique invariant measure for  $f$ .

It remains to show that  $\underline{d}_\mu(x) \geq \beta$ . Recall that  $h = \lim h_n$  is the distribution function of  $\mu$ .

LEMMA 2.2.  *$h$  is Hölder continuous with the exponent  $\beta$ .*

*Proof.* It suffices to show that  $|h_n(x) - h_n(y)| \leq C|x - y|^\beta$  for all  $x, y \in S^1$  and  $n \geq 0$ . We will prove by induction that for all  $n \geq 0$ ,  $h_n$  has the following properties:

- (i)  $|h_n(x) - h_n(y)| \leq |x - y|^\beta$  for all  $x, y \in S^1$  with  $|x - y| \leq s_n$ ;
- (ii)  $|h_n(x) - h_n(y)| \leq (4 - 2^{-n})|x - y|^\beta$  for all  $x, y$  with  $|x - y| \geq s_n$ .

For  $h_0 = \text{Id}$  this holds true. We now show that  $h_n$  has properties (i) and (ii) provided that all  $h_i$  with  $i < n$  do. If  $|x - y| \leq r_n$ , then

$$\begin{aligned} |h_n(x) - h_n(y)| &\leq \frac{s_n}{d_n} M_{n-1} |x - y| \leq \frac{r_n^{\beta+n-1}}{r_n m_{n-1}} M_{n-1} |x - y| \\ &\leq \frac{M_{n-1}}{m_{n-1}} |x - y|^{\beta+n-1} \leq |x - y|^\beta \end{aligned}$$

since  $|x - y|^{1/n} \leq r_n^{1/n} \leq m_{n-1}/M_{n-1}$  by the choice of  $d_n$ . If  $r_n \leq |x - y| \leq s_n$  then

$$|h_n(x) - h_n(y)| \leq s_n(M_{n-1} + 1) = r_n^{\beta+n-1}(M_{n-1} + 1) \leq |x - y|^\beta$$

since  $r_n^{1/n} \leq (M_{n-1} + 1)$  again by the choice of  $d_n$ , and  $|A_n(x) - A_n(y)| \leq s_n$  if  $|x - y| \leq s_n$ . So we conclude that  $h_n$  has property (i). If  $s_n \leq |x - y| \leq s_{n-1}$  then

$$|h_n(x) - h_n(y)| \leq 2s_n + |h_{n-1}(x) - h_{n-1}(y)| \leq 2s_n + |x - y|^\beta \leq 3|x - y|^\beta.$$

If  $s_{n-1} \leq |x - y|$  then

$$|h_n(x) - h_n(y)| \leq 2s_n + |h_{n-1}(x) - h_{n-1}(y)| \leq 2s_n + (4 - 2^{-n+1})|x - y|^\beta \leq (4 - 2^n)|x - y|^\beta$$

since  $2s_n \leq 2^{-n}s_{n-1}$  by the choice of  $l_n$ .

So we conclude that  $h_n$  also has property (ii). This completes the proof of the lemma.  $\square$

Lemma 2.2 implies that  $\underline{d}_\mu(x) \geq \beta$  for all  $x \in S^1$ . This completes the proof of Proposition 2.2.  $\square$

Now we will prove Theorem 2.1 using the construction in Proposition 2.1.

*Proof.* Let  $\tilde{f} \in D'_f$ . In any  $C^r$ -neighborhood of  $\tilde{f}$  there exists a  $C^\infty$  diffeomorphism  $f_*$  with a Diophantine rotation number. Such diffeomorphisms are known to be  $C^\infty$ -conjugate to corresponding rotations, i.e.  $f_* = h_*^{-1}R_{\tau_*}h_*$ , where  $h_*$  is a  $C^\infty$  circle diffeomorphism (see [5]).  $f_*$  can be used as a starting point for a sequence of iterations  $f_n$  constructed as in the proof of Proposition 2.1. Then the sequence  $f_n$  converges in  $C^r$ -topology to some diffeomorphism  $f$  which can be made as close to  $f_*$  as we wish and satisfies the following condition. For any  $n > 0$  there exists a set  $E_n$  with  $\mu(S^1 \setminus E_n) < 2^{-n+1}$  and positive numbers  $r_n > \tilde{r}_n$  such that

$$\begin{aligned} \frac{\log \mu(B(x, r_n))}{\log r_n} &< \frac{2}{n} \quad \text{for any } x \in E_n, \\ \frac{\log \mu(B(x, \tilde{r}_n))}{\log \tilde{r}_n} &> 1 - \frac{2}{n} \quad \text{for any } x \in S^1, \end{aligned}$$

where  $\mu$  is the unique invariant measure for  $f$ , and  $r_n, \tilde{r}_n \leq n^{-1}$ . So we see that  $D'_f$  contains a dense subset  $Z$  of diffeomorphisms satisfying the above condition.

For any diffeomorphism  $f \in Z$  we can construct a sequence of its neighborhoods,  $\{V_n^f\}_{n=1}^\infty$ , such that any uniquely ergodic diffeomorphism  $g$  in  $V_n^f$  satisfies the condition

$$\begin{aligned} \frac{\log \nu(B(x, r_n))}{\log r_n} &< \frac{3}{n} \quad \text{for any } x \in E_n, \\ \frac{\log \nu(B(x, \tilde{r}_n))}{\log \tilde{r}_n} &> 1 - \frac{3}{n} \quad \text{for any } x \in S^1, \end{aligned}$$



where  $\nu$  is the unique invariant measure for  $g$ ,  $E_n, r_n$  and  $\tilde{r}_n$  are the same as for  $f$ , and  $\nu(S^1 \setminus E_n) < 2^{-n+2}$ . Indeed, if  $f$  and  $g$  are sufficiently close in  $C^0$ -topology, their invariant measures are sufficiently close in the weak topology.

Let  $Y_0^r = \bigcap_{n=1}^\infty \bigcup_{f \in Z} V_n^f$ . Then  $Y_0^r \cap \overline{D_I^r}$  and  $Y_0^r \cap D_I^r$  are residual subsets of  $\overline{D_I^r}$  and  $D_I^r$ , respectively.

Any uniquely ergodic diffeomorphism  $g$  in  $Y_0^r$  satisfies the above condition for some sequence of scales  $\{r_n\}$  and  $\{\tilde{r}_n\}$  which converge to zero. It follows that  $\underline{d}_\nu(x) = 0$ ,  $\overline{d}_\nu(x) = 1$  for  $\nu$ -a.a.  $x \in S^1$ .

It is easy to see that the set  $E_n$  can be covered by  $1/s_n$  balls of radius  $r_n$  (recall that  $s_n$  is the period of the function  $a_n$ ; see the proof of Proposition 2.1). Since  $\log s_n / \log r_n \rightarrow 0$  as  $n \rightarrow \infty$  we see that  $\underline{\dim}_B(\bigcap_{n=k}^\infty E_n) = 0$  for any  $k > 0$ . Since  $\nu(\bigcap_{n=k}^\infty E_n) > 1 - 2^{-n+3} \rightarrow 1$  we conclude that  $\underline{\dim}_B \nu = 0$ .

On the other hand, since  $\nu(B(x, \tilde{r}_n)) < \tilde{r}_n^{1-3n^{-1}}$  for any  $x \in S^1$  the minimal number  $N$  of balls of radius needed to cover a set of  $\nu$  measure  $1 - \varepsilon$  is at least  $(1 - \varepsilon)\tilde{r}_n^{-(1-3n^{-1})}$ . Hence

$$\frac{\log N}{-\log \tilde{r}_n} \geq 1 - \frac{3}{n} + \frac{\log(1 - \varepsilon)}{-\log \tilde{r}_n} \xrightarrow{n \rightarrow \infty} 1$$

and we conclude that  $\overline{\dim}_B \nu \geq 1$ . Since

$$0 \leq \dim_H \nu \leq \underline{\dim}_B \nu \leq \overline{\dim}_B \nu \leq 1$$

for any finite measure on  $S^1$  we see that

$$\dim_H \nu = \underline{\dim}_B \nu = 0 \quad \text{and} \quad \overline{\dim}_B \nu = 1.$$

We conclude that any uniquely ergodic diffeomorphism  $g$  in  $Y_0^r$  lies in  $Y^r$ . Since the set of the diffeomorphisms in  $\overline{D_I^r}$  which are not uniquely ergodic is of the first category, we conclude that  $Y^r$  is a residual subset in both  $D_I^r$  and  $\overline{D_I^r}$ .  $\square$

The proof of Theorem 2.2 uses Proposition 2.2 and follows the corresponding steps of the proof of Theorem 2.1 almost identically. The lower bound for the Hausdorff dimension of the measure is provided in this case by the following fact (see [8]): if  $\underline{d}_\mu(x) \geq \beta$  for  $\mu$ -a.a.  $x$  then  $\dim_H \mu \geq \beta$ .

We now complete the section with the proof of Corollary 2.1.

*Proof.* It suffices to show that for any open and dense subset  $U \subset D_I^r$  there exists a residual subset  $T \subset [0, 1]$  such that for any  $\tau \in T$  the intersection  $U \cap D_\tau^r$  is open and dense in  $D_\tau^r$ . Since  $U \cap D_\tau^r$  is open in the induced topology we only need to check whether it is dense.

Let us suppose that there exists a subset  $S \subset [0, 1]$  of the second category such that for any  $\tau \in S$  the intersection  $U \cap D_\tau^r$  is not dense in  $D_\tau^r$ . In other words, for any  $\tau \in S$  there exist  $f_\tau \in D_\tau^r$  and  $r_\tau > 0$  such that  $B(f_\tau, r_\tau) \cap D_\tau^r \cap U = \emptyset$ , where  $B(f_\tau, r_\tau)$  is the ball in  $D^r$  centered at  $f_\tau$  of radius  $r_\tau$ . Then for some  $\varepsilon > 0$  there exists  $S_1 \subset S$  of the second category in  $[0, 1]$  such that  $r_\tau > 3\varepsilon$  for all  $\tau \in S_1$ . Since  $D_I^r$  is second countable there exists a countable  $\varepsilon$ -spanning set  $\{g_n\} \subset D_I^r$ . Then for some  $i > 0$  there exists  $S_2 \subset S_1$  of the second category in  $[0, 1]$  such that  $f_\tau \in B(g_i, \varepsilon)$  for all  $\tau \in S_2$ . Set  $I = \tau(B(g_i, \varepsilon))$  and by  $I_u = \tau(B(g_i, \varepsilon)) \cap U$ , where  $\tau : D^r \rightarrow [0, 1]$  is the rotation number function.

We obtain  $S_2 \subset I$  and  $S_2 \cap I_u = \emptyset$  since for all  $\tau \in S_2$  we have  $B(f_\tau, 3\epsilon) \cap D'_\tau \cap U = \emptyset$  and  $f_\tau \in B(g_i, \epsilon)$  whence  $B(g_i, \epsilon) \cap D'_\tau \cap U = \emptyset$ . We note that  $I_u$  is open in  $I \setminus \mathbb{Q}$  and  $I \setminus I_u$  is of the second category in  $I$  since  $S_2$  is. We conclude that  $I \setminus I_u$  has a non-empty interior. It follows that there exists an interval  $I_s \subset I$  such that  $I_s \cap I_u = \emptyset$ . This implies that the set  $\tau^{-1}(I_s) \cap B(g_i, \epsilon) \cap D'_\tau$  is open in  $D'_\tau$  and does not intersect  $U$ . This contradicts to the fact that  $U$  is dense and completes the proof of the corollary.  $\square$

3. Circle homeomorphisms

In the previous section we have shown that for a circle diffeomorphism we can make the lower pointwise dimension of its invariant measure  $\mu$  equal to any number between 0 and 1. We do not know whether there exists a circle diffeomorphism such that  $\bar{d}_\mu(x) = \gamma < 1$  for  $\mu$ -a.e.  $x \in S^1$ . However, we can obtain such pinching in the case of Hölder circle homeomorphisms. Moreover, we can construct Hölder circle homeomorphisms such that the pointwise dimension exists almost everywhere and is equal to a given number  $\alpha$ ,  $0 < \alpha < 1$ . We show that such homeomorphisms are dense in the set of all circle homeomorphisms with a given irrational rotation number.

Denote by  $H_\tau$ ,  $\tau \in [0, 1] \setminus \mathbb{Q}$ , the set of all homeomorphisms of  $S^1$  with rotation number  $\tau$ .

THEOREM 3.1.

- (1) For any  $\beta, \gamma$ ,  $0 < \beta < \gamma \leq 1$ , the set of all Hölder homeomorphisms whose invariant measure has lower pointwise dimension equal to  $\beta$  and upper pointwise dimension equal to  $\gamma$  for a.e.  $x \in S^1$  is everywhere dense in  $H_\tau$ .
- (2) For any  $\alpha \in (0, 1]$  the set of all Hölder homeomorphisms whose invariant measure has pointwise dimension  $\alpha$  for a.e.  $x \in S^1$  is everywhere dense in  $H_\tau$ .

The proof of Theorem 3.1 is based on the following proposition.

PROPOSITION 3.1.

- (1) For any  $\beta, \gamma$  such that  $0 < \beta < \gamma \leq 1$ , the set of all Borel probability measures  $\mu$  on  $S^1$  such that  $\underline{d}_\mu(x) = \beta$  and  $\bar{d}_\mu(x) = \gamma$  for  $\mu$ -a.e.  $x \in S^1$  is everywhere dense (in the weak topology) in the set of all Borel probability measures on  $S^1$ .
- (2) For any  $\alpha \in (0, 1]$  the set of all Borel probability measures  $\nu$  on  $S^1$  such that  $d_\nu(x) = \alpha$  for  $\nu$ -a.e.  $x \in S^1$  is everywhere dense in the set of all Borel probability measures on  $S^1$ .

*Proof of Proposition 3.1.* To obtain measures with desired properties on  $S^1$  we first construct their counterparts on the symbolic space

$$\Omega_2 = \{\omega = (\omega_0\omega_1 \dots) : \omega_i \in \{0, 1\}, i \in \mathbb{N}_0\}.$$

Then we use the binary coding of the unit interval to carry the measures to  $S^1$ .

- (1) Let us fix  $\beta$  and  $\gamma$  such that  $0 < \beta < \gamma \leq 1$  and take the numbers  $p, q, \tilde{p}, \tilde{q}$  such that

$$\begin{aligned} 0 < p \leq q < 1, \quad p + q = 1, \quad p \log p + q \log q = \beta \log \frac{1}{2}, \\ 0 < \tilde{p} \leq \tilde{q} < 1, \quad \tilde{p} + \tilde{q} = 1, \quad \tilde{p} \log \tilde{p} + \tilde{q} \log \tilde{q} = \gamma \log \frac{1}{2}. \end{aligned}$$

Let

$$\begin{aligned} s_n^0 &= p, & s_n^1 &= q & \text{for } 2^{(2k)!} \leq n < 2^{(2k+1)!} \\ s_n^0 &= \tilde{p}, & s_n^1 &= \tilde{q} & \text{for } 2^{(2k+1)!} \leq n < 2^{(2k+2)!}. \end{aligned}$$

For any cylinder  $C_{\omega_m \dots \omega_n}$  we set  $\hat{\mu}(C_{\omega_m \dots \omega_n}) = \prod_{i=m}^n s_i^{\omega_i}$ . Consider the independent random variables

$$\xi_i = \begin{cases} \log p, & \text{if } \omega_i = 0 \text{ and } 2^{(2k)!} \leq i < 2^{(2k+1)!} \\ \log q, & \text{if } \omega_i = 1 \text{ and } 2^{(2k)!} \leq i < 2^{(2k+1)!} \\ \log \tilde{p}, & \text{if } \omega_i = 0 \text{ and } 2^{(2k+1)!} \leq i < 2^{(2k+2)!} \\ \log \tilde{q}, & \text{if } \omega_i = 1 \text{ and } 2^{(2k+1)!} \leq i < 2^{(2k+2)!}. \end{cases}$$

Denote the expectation and the dispersion of  $\xi_i$  by  $A_i$  and  $D_i$ , respectively. We have that

$$\begin{aligned} A_i &= \int_{\Omega_2} \xi_i d\mu = \begin{cases} \beta \log \frac{1}{2} & \text{if } 2^{(2k)!} \leq i < 2^{(2k+1)!} \\ \gamma \log \frac{1}{2} & \text{if } 2^{(2k+1)!} \leq i < 2^{(2k+2)!} \end{cases} \\ D_i &= \int_{\Omega_2} |\xi_i - A_i|^2 d\mu. \end{aligned}$$

One can see that  $D_i$  is bounded by a constant independent of  $i$ . Therefore  $\sum_{i=0}^\infty i^{-2} D_i < \infty$ , and the Law of Large Numbers yields

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} \xi_i(\omega) - \frac{1}{n} \sum_{i=0}^{n-1} A_i \right) = 0 \quad \text{for } \hat{\mu}\text{-a.e. } \omega \in \Omega_2,$$

in particular,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} \xi_i(\omega) \right) &= \liminf_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} A_i \right) = \beta \log \frac{1}{2} \\ \limsup_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} \xi_i(\omega) \right) &= \limsup_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{i=0}^{n-1} A_i \right) = \gamma \log \frac{1}{2} \end{aligned}$$

for  $\hat{\mu}$ -a.e.  $\omega \in \Omega_2$ . It follows that for  $\hat{\mu}$ -a.e.  $\omega \in \Omega_2$

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{n^{-1} \log \hat{\mu}(C_{\omega_0 \dots \omega_{n-1}})}{\log(1/2)} &= \liminf_{n \rightarrow \infty} \frac{n^{-1} \sum_{i=0}^{n-1} \xi_i(\omega)}{\log(1/2)} = \beta, \\ \limsup_{n \rightarrow \infty} \frac{n^{-1} \log \hat{\mu}(C_{\omega_0 \dots \omega_{n-1}})}{\log(1/2)} &= \gamma. \end{aligned}$$

Let us consider the binary coding  $\phi : \Omega_2 \rightarrow [0, 1]$  of the interval  $[0, 1]$ . Recall that each number has at most two binary expansions and any irrational number has exactly one.

Fix a measure  $\kappa_0$  on  $S^1$ . Consider a measure  $\kappa$  with no atoms which is positive on open intervals and close to  $\kappa_0$  in the weak topology. Let  $\hat{\kappa}$  be its pull back to  $\Omega_2$  by  $\phi$ .

Fix  $n \in \mathbb{N}$ . For any cylinder  $C_{\omega_0 \dots \omega_m}$  set

$$\hat{\kappa}_j(C_{\omega_0 \dots \omega_m}) = \begin{cases} \hat{\kappa}(C_{\omega_0 \dots \omega_m}), & \text{if } m < j \\ \hat{\kappa}(C_{\omega_0 \dots \omega_{n-1}}) \hat{\mu}(C_{\omega_n \dots \omega_m}), & \text{if } m \geq j \end{cases}$$

where  $\hat{\mu}$  is the measure constructed above. It is easy to see that for any  $j$  we have that

$$\liminf_{n \rightarrow \infty} \frac{n^{-1} \log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{\log(1/2)} = \beta, \quad \limsup_{n \rightarrow \infty} \frac{n^{-1} \log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{\log(1/2)} = \gamma$$

for  $\hat{\kappa}_j$ -a.e.  $\omega \in \Omega_2$ .

Let us denote by  $\kappa_j$  the push forward of  $\hat{\kappa}_j$  to  $[0, 1]$  by  $\phi$ . Clearly, the measure  $\kappa_j$  is close to  $\kappa$  for large  $j$ , positive on open intervals and has no atoms. To complete the proof of the second statement of the proposition it remains to prove the following lemma.

LEMMA 3.1.  $\underline{d}_{\kappa_j}(x) = \beta$  and  $\overline{d}_{\kappa_j} = \gamma$  for  $\kappa_j$ -a.e.  $x \in S^1$ .

*Proof.* Note that  $\phi(C_{\omega_0 \dots \omega_{n-1}})$  is one of  $2^n$  closed binary intervals of length  $2^{-n}$ . So we see that  $\phi^{-1}(B(x, 2^{-n})) \supset C_{\omega_0 \dots \omega_{n-1}}$  for any  $x \in S^1$ , where  $\phi(\omega_0 \omega_1 \dots) = x$ . It follows that, for  $\kappa_j$ -a.e.  $x \in S^1$

$$\begin{aligned} \underline{d}_{\kappa_j}(x) &= \liminf_{n \rightarrow \infty} \frac{\log \kappa_j(B(x, 2^{-n}))}{\log 2^{-n}} \leq \lim_{n \rightarrow \infty} \frac{\log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{n \log(1/2)} \\ &= \lim_{n \rightarrow \infty} \left( \frac{n + 2 \log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{n(n + 2) \log(1/2)} \right) = \beta, \\ \overline{d}_{\kappa_j}(x) &= \limsup_{n \rightarrow \infty} \frac{\log \kappa_j(B(x, 2^{-n}))}{\log 2^{-n}} \leq \lim_{n \rightarrow \infty} \frac{\log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{n \log(1/2)} = \gamma. \end{aligned}$$

To obtain the estimates below we introduce the following sets:

$$B_k = \bigcup_{i=1}^{2^k} \left[ \frac{i}{2^k} - \frac{1}{2^{k+\lceil \sqrt{k} \rceil}}, \frac{i}{2^k} + \frac{1}{2^{k+\lceil \sqrt{k} \rceil}} \right] \subset S^1$$

and

$$G_m = S^1 \setminus \left( \bigcup_{k=m}^{\infty} B_k \right).$$

It is easy to see that for any  $x \in G_m$  and any  $n > m$ , we have

$$\phi^{-1}(B(x, 2^{-(n+\lceil \sqrt{n} \rceil)})) \subset C_{\omega_0 \dots \omega_{n-1}}.$$

Hence, for  $\kappa_j$ -a.e.  $x \in G_m$ ,

$$\begin{aligned} \underline{d}_{\kappa_j}(x) &= \liminf_{n \rightarrow \infty} \frac{\log \kappa_j(B(x, 2^{-(n+\lceil \sqrt{n} \rceil)}))}{\log 2^{-(n+\lceil \sqrt{n} \rceil)}} \geq \lim_{n \rightarrow \infty} \frac{\log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{(n + \lceil \sqrt{n} \rceil) \log(1/2)} \\ &= \lim_{n \rightarrow \infty} \left( \frac{n}{n + \lceil \sqrt{n} \rceil} \frac{\log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{n \log(1/2)} \right) = \beta, \\ \overline{d}_{\kappa_j}(x) &= \limsup_{n \rightarrow \infty} \frac{\log \kappa_j(B(x, 2^{-(n+\lceil \sqrt{n} \rceil)}))}{\log 2^{-(n+\lceil \sqrt{n} \rceil)}} \geq \lim_{n \rightarrow \infty} \frac{\log \hat{\kappa}_j(C_{\omega_0 \dots \omega_{n-1}})}{(n + \lceil \sqrt{n} \rceil) \log(1/2)} = \gamma. \end{aligned}$$

For any  $k > j$  we observe that  $\kappa_j(B_k) \leq 2q\sqrt{n}$ , where  $q < 1$  is from the construction of the measure  $\hat{\nu}$ . Hence,  $\kappa_j(G_m) \nearrow 1$  as  $m \rightarrow \infty$ . It follows that  $\underline{d}_{\kappa_j} \geq \beta$  and  $\overline{d}_{\kappa_j} \geq \gamma$  for  $\kappa_j$ -a.e.  $x \in S^1$ , and this completes the proof of the lemma.  $\square$

This completes the proof of the first statement of Proposition 3.1.

(2) Let us fix  $\alpha \in (0, 1]$  and take the numbers  $p$  and  $q$  such that

$$0 < p \leq q < 1, \quad p + q = 1 \quad \text{and} \quad p \log p + q \log q = \alpha \log \frac{1}{2}.$$

Let us consider the Bernoulli measure  $\nu = \nu(p, q)$  on  $\Omega_2$  which is defined as follows. For any cylinder

$$C_{\omega_m \dots \omega_n} = \{\omega' \in \Omega_2 : \omega'_i = \omega_i, m \leq i \leq n\},$$

$$\hat{\nu}(C_{\omega_m \dots \omega_n}) = \prod_{i=m}^n s_i^{\omega_i}, \quad \text{where } s_i^0 = p \text{ and } s_i^1 = q.$$

This measure is ergodic with respect to the shift  $\sigma$ . Clearly, it has no atoms and is positive on any cylinder.

Consider the function

$$g(\omega) = \begin{cases} \log p, & \text{if } \omega_0 = 0 \\ \log q, & \text{if } \omega_0 = 1. \end{cases}$$

By the Birkhoff Ergodic Theorem

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\sigma^i(\omega)) = \int_{\Omega_2} g \, d\hat{\nu} = p \log p + q \log q \quad \text{for } \hat{\nu}\text{-a.e. } \omega \in \Omega_2.$$

This implies that for  $\hat{\nu}$ -a.e.  $\omega \in \Omega_2$ ,

$$\lim_{n \rightarrow \infty} \frac{n^{-1} \log \hat{\nu}(C_{\omega_0 \dots \omega_{n-1}})}{\log(1/2)} = \frac{p \log p + q \log q}{\log(1/2)} = \alpha.$$

We now use the measure  $\hat{\nu}$  to modify a given measure  $\kappa_0$  in the same way as we used  $\hat{\mu}$  in the proof of the first statement, and the rest of the proof follows similarly.  $\square$

Now we will prove Theorem 3.1.

*Proof of Theorem 3.1.* Fix an irrational rotation number  $\tau$  and consider a diffeomorphism  $\tilde{f} \in H_\tau$ . In any neighborhood of  $\tilde{f}$  there exists a  $C^2$  circle diffeomorphism  $f_*$  with an irrational rotation number. By the Denjoy Theorem it is conjugate to the corresponding rotation:  $f_* = h_*^{-1} \circ R_\tau \circ h_*$ . Consider the homeomorphism  $f_0 = h_*^{-1} \circ R_\tau \circ h_*$ . It is close to  $\tilde{f}$  and has the same rotation number.

(2) Let  $\kappa_0$  be the invariant measure for  $f_0$ . Consider a sequence of measures  $\kappa_j$  without atoms and positive on open intervals which have pointwise dimension equal to  $\alpha$  for  $\kappa_j$ -a.a.  $x \in S^1$  and weakly converge to  $\kappa$  (constructed as in Proposition 3.1). Let  $h_n$  be the distribution function of  $\kappa_j$  and  $f_n = h_n^{-1} \circ R_\tau \circ h_n$ . Then it is easy to see that  $f_n$  converge uniformly to  $f_0$  and  $f_n^{-1}$  converge uniformly to  $f_0^{-1}$ .

LEMMA 3.2. *The homeomorphisms  $f_n$  constructed above are Hölder continuous with Hölder exponent  $\log q/2 \log p$ .*

*Proof.* Let  $A$  and  $B$  be binary intervals,  $|A| = 2^{-m}$ ,  $|B| = 2^{-k}$ , such that  $\kappa_j(A) \leq \kappa_j(B)$ . We will show that  $m/k \geq \log q/2 \log p$ , i.e.  $|A| \leq |B|^{(\log q)/(2 \log p)}$ .

Recall that  $\phi^{-1}(A) = C_{\omega_0 \dots \omega_{m-1}}$  and  $\phi^{-1}(B) = C_{\omega'_0 \dots \omega'_{k-1}}$  for some  $(\omega_0 \dots \omega_{m-1})$  and  $(\omega'_0 \dots \omega'_{k-1})$  (up to countably many elements). We can assume that  $m, k > n$ . Then

$$\hat{\kappa}_j(C_{\omega_0 \dots \omega_m}) = \hat{\kappa}(C_{\omega_0 \dots \omega_{n-1}}) \prod_{i=n}^{m-1} s_i^{\omega_i} \leq \hat{\kappa}(C_{\omega'_0 \dots \omega'_{n-1}}) \prod_{j=n}^{k-1} s_j^{\omega'_j} = \hat{\kappa}_j(C_{\omega'_0 \dots \omega'_m}).$$

Let

$$M_n = \max \frac{\hat{\kappa}(C_{\omega_0 \dots \omega_n})}{\hat{\kappa}(C_{\omega'_0 \dots \omega'_n})},$$

where maximum is taken over all cylinders of length  $n + 1$ . The ratio  $m/k$  is the smallest when  $s_i^{\omega_i} = p, i = n, \dots, m - 1$  and  $s_j^{\omega'_j} = q, j = n, \dots, k - 1$ . Therefore  $p^{m-n} \leq Mq^{k-n}$  and

$$\frac{m}{k} \geq \frac{\log q}{\log p} + \frac{\log M + n \log(p/q)}{k \log p} \geq \frac{\log q}{2 \log p}$$

if  $k$  is big enough.

Let  $I$  be an interval,  $A \subset I$  be a binary interval (i.e. the image of a cylinder in  $\Omega_2$  under  $\phi$ ) of the largest possible length and  $B \supset f_n(I)$  a binary interval of the smallest possible length. Then  $\kappa_j(A) = \kappa_j(f_n(A)) \leq \kappa_j(f_n(I)) \leq \kappa_j(B)$ . Hence

$$\begin{aligned} |I| &\leq 2|A| \leq 2|B|^{(\log q)/(2 \log p)} \leq 2(2|f_n(I)|)^{(\log q)/(2 \log p)} \\ &= 2^{(\log q)/(2 \log p)+1} |f_n(I)|^{(\log q)/(2 \log p)}. \end{aligned}$$

The same argument shows that  $|f_n(I)| \leq 2^{(\log q)/(2 \log p)+1} |I|^{(\log q)/(2 \log p)}$ . □

This completes the proof of the second part of the theorem. The first part can be proven similarly. □

#### 4. Analytic circle diffeomorphisms

Let us fix an annulus  $A \subset \mathbb{C}$  containing  $S^1$  and denote by  $D^\omega = D^\omega(A) \subset \text{Diff}^\omega(S^1)$  the set of all orientation-preserving circle diffeomorphisms  $f$  such that  $f$  and  $f^{-1}$  extend to analytic functions on  $A$ . We endow  $D^\omega$  with the topology of uniform convergence on compact subsets of  $A$ . Denote by  $D_I^\omega$  the subset of  $D^\omega$  consisting of all diffeomorphisms with irrational rotation number.

Let  $Y^\omega$  be the subset of  $D_I^\omega$  consisting of diffeomorphisms  $f$  such that:

- (1)  $\underline{d}_\mu(x) = 0$  and  $\overline{d}_\mu(x) = 1$  for  $\mu$ -a.a.  $x \in S^1$ ;
- (2)  $\dim_H \mu = \underline{\dim}_B \mu = 0$  and  $\overline{\dim}_B \mu = 1$ ,

where  $\mu$  is the invariant measure for  $f$ . The following statements are analytic counterparts of Theorem 2.1 and Corollary 2.1.

**THEOREM 4.1.**  $Y^\omega$  is a residual subset of both  $D_I^\omega$  and  $\overline{D_I^\omega}$ .

The proof of Theorem 4.1 is based on Propositions 4.2 and 4.3 below.

Let  $D_\tau^\omega$  be the set of all diffeomorphisms in  $D^\omega$  with rotation number  $\tau$ .

**COROLLARY 4.1.** *There exists a set  $T^\omega \subset [0, 1] \setminus \mathbb{Q}$  which is a residual subset of  $[0, 1]$  such that for any  $\tau \in T^\omega, Y^\omega \cap D_\tau^\omega$  is a residual subset of  $D_\tau^\omega$ .*

*Proof.* The proof is identical to the proof of Corollary 2.1. □

**Remark 4.1.** The set  $T^\omega$  has zero Lebesgue measure and zero Hausdorff dimension (compare to Remark 2.1).

We have a natural projection  $\pi : \mathbb{R} \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$ . This provides a lift of a diffeomorphism  $f : S^1 \rightarrow S^1$  to a diffeomorphism  $F : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f \circ \pi = \pi \circ F$ .

Let  $f$  be an analytic orientation-preserving diffeomorphism. We say that  $f$  satisfies the property  $(\star)$  if for any  $\alpha \in [0, 1]$ , no power of the diffeomorphism

$$f_\alpha = f + \alpha \pmod{1}$$

is the identity map.

The following proposition proves the existence of diffeomorphisms satisfying the property  $(\star)$ .

**PROPOSITION 4.1.** *Let  $f : S^1 \rightarrow S^1$  be an orientation-preserving diffeomorphism such that it is not a rotation and its lift  $F : \mathbb{R} \rightarrow \mathbb{R}$  extends to an entire function. Then  $f$  satisfies the property  $(\star)$ .*

*Proof.* Suppose  $f_\alpha^q = \text{Id}$  for some  $q \in \mathbb{N}$  and  $\alpha \in [0, 1]$ . Then  $F_\alpha^q = \text{Id} + p$  on  $\mathbb{C}$  for some  $p \in \mathbb{Z}$ . This implies that  $F : \mathbb{C} \rightarrow \mathbb{C}$  is a bijection. Since  $F$  is entire it must be a linear function. It follows that the diffeomorphism  $f$  is a rotation.  $\square$

The following proposition shows that property  $(\star)$  diffeomorphisms are typical.

**PROPOSITION 4.2.** *The diffeomorphisms in  $D_I^\omega$  satisfying the property  $(\star)$  form a residual subset of  $\overline{D_I^\omega}$ .*

The following proof was communicated to us by Keith Burns.

*Proof.* We will prove that the set

$$\{f \in D_I^\omega : f_\alpha^n \neq \text{Id} \text{ for all } \alpha \in [0, 1] \text{ and } n \geq 1\}$$

is a residual subset of  $\overline{D_I^\omega}$ . It suffices to show that for every  $n \geq 1$  the set

$$G_n = \{f \in D_I^\omega : f_\alpha^n \neq \text{Id} \text{ for all } \alpha \in [0, 1]\}$$

is open and dense in  $\overline{D_I^\omega}$ .

Fix  $n \geq 1$ . It is easy to see that the complement of  $G_n$  is closed so it is enough to check that  $G_n$  is dense in  $\overline{D_I^\omega}$ . Let  $U \subset \overline{D_I^\omega}$  be an open set. We will show that  $U \cap G_n \neq \emptyset$ . Let us take a diffeomorphism  $f \in U$  with irrational rotation number and suppose that  $f \notin G_n$ .

Let  $F$  be a lift of  $f$ , then  $F_\alpha = F + \alpha$  is a lift of  $f_\alpha$ . The equality  $f_\alpha^n = \text{Id}$  is equivalent to  $F_\alpha^n = \text{Id} + p$  for some  $p \in \mathbb{Z}$ . Since  $F_\alpha^n(x)$  is increasing in  $\alpha$  there are only finitely many values of  $\alpha$  in  $[0, 1]$  for which  $f_\alpha^n = \text{Id}$ . Let us denote these values by  $\alpha_1, \dots, \alpha_k$ .

Let  $E = [0, 1] \setminus (I_1 \cup \dots \cup I_k)$ , where  $I_j, j = 1, \dots, k$ , are open intervals centered at  $\alpha_i$  of length  $(\max(2, \sup |f'|))^{-(n+1)}$ . Since  $f_\alpha^n \neq \text{Id}$  for any  $\alpha \in E$  there exists a neighborhood  $U_0 \subset U$  of  $f$  such that  $g_\alpha^n \neq \text{Id}$  for any  $g \in U_0$  and any  $\alpha \in E$ .

**LEMMA 4.1.** *Let  $f \in D_I^\omega$  be such that  $F_\alpha = \text{Id} + p$  for some  $p \in \mathbb{Z}$  and  $\alpha \in [0, 1]$ . Then in any neighborhood of  $f$  there exists  $\tilde{f} \in D_I^\omega$  such that for its lift  $\tilde{F}$ ,*

$$\tilde{F}_\alpha^n(x') < x' + p \quad \text{and} \quad \tilde{F}_\alpha^n(x'') > x'' + p$$

for some  $x', x'' \in \mathbb{R}$ .

*Proof.* Let us fix  $x \in S^1$  and consider its orbit  $x, fx, \dots, f^m x$ , where  $m + 1$  is the minimal period of  $x$ . There exists an analytic flow  $\phi^t$  on  $S^1$  for which  $x, fx, \dots, f^m x$  are repelling fixed points. In other words, for  $t > 0, 1 \leq i \leq m$ , and for all  $y$  sufficiently close to  $f^i x$  we have  $\Phi^t y > y$  if  $y > x$  and  $\Phi^t y < y$  if  $y < x$ , where  $\Phi^t$  is the lift of  $\phi^t$  such that  $\Phi^0 = \text{Id}$ . It is easy to see that if  $x' < x$  and  $x'' > x$  are sufficiently close to  $x$  and if  $t > 0$  is small then  $\tilde{f} = f \circ \phi^t$  satisfies the conditions of the lemma. We note that since  $f$  has irrational rotation number,  $\tilde{f}$  can be also chosen to have an irrational rotation number and can be made as close to  $f$  as we wish.  $\square$

Since  $F_{\alpha_1} = \text{Id} + p_1$  for some  $p_1 \in \mathbb{Z}$ , Lemma 4.1 implies that there exists  $\tilde{f} \in U_0$  such that  $\tilde{F}_{\alpha_1}^n(x') < x' + p_1$  and  $\tilde{F}_{\alpha_1}^n(x'') > x'' + p_1$  for some  $x', x'' \in \mathbb{R}$ . It follows that for  $\alpha \in I_1$  we have  $\tilde{F}_{\alpha}^n(x') < x' + p_1$  if  $\alpha < \alpha_1$ , and  $\tilde{F}_{\alpha}^n(x'') > x'' + p_1$  if  $\alpha > \alpha_1$ .

If  $\tilde{f}$  is chosen close to  $f$  then  $|\tilde{F}_{\alpha}(x) - F_{\alpha_1}(x)| < |I_1|$  for any  $\alpha \in I_1$  and  $x \in \mathbb{R}$ . So by the choice of the length  $|I_1|$  it follows that

$$|\tilde{F}_{\alpha}^n(x) - (x + p_1)| = |\tilde{F}_{\alpha}^n(x) - F_{\alpha_1}^n(x)| < 1$$

and therefore

$$x + p_1 - 1 < \tilde{F}_{\alpha}^n(x) < x + p_1 + 1$$

for any  $\alpha \in I_1$  and  $x \in \mathbb{R}$ . So we conclude that  $\tilde{f}_{\alpha} \neq \text{Id}$  for  $\alpha \in I_1 \cup E$ .

We can choose a neighborhood  $U_1 \subset U_0$  of  $\tilde{f}$  such that  $g_{\alpha}^n \neq \text{Id}$  for any  $g \in U_1$  and any  $\alpha \in E \cup I_1$ . The proposition now follows by consecutive application of Lemma 4.1.  $\square$

**PROPOSITION 4.3.** *Let  $f \in D_1^{\varphi}$  satisfy the property  $(\star)$ . Then for any  $\delta > 0$  there exists  $\alpha_*$ ,  $0 < \alpha_* < \delta$ , such that the diffeomorphism  $f_{\alpha_*} = f + \alpha_* \pmod{1}$  has irrational rotation number and for its unique invariant measure  $\mu$ ,  $\underline{d}_{\mu}(x) = 0$  and  $\bar{d}_{\mu}(x) = 1$  for  $\mu$ -a.a.  $x \in S^1$ .*

*Proof.* Our construction uses the method of constructing analytic circle diffeomorphisms with singular conjugacy described in [6].

Consider the family of analytic circle diffeomorphisms

$$f_{\alpha} = f + \alpha \pmod{1}, \quad \text{where } 0 < \alpha < \delta.$$

Denote by  $\tau(\alpha)$  the rotation number of  $f_{\alpha}$ . This family has the following properties:

- (a)  $\tau(\alpha)$  is non-decreasing in  $\alpha$ ;
- (b)  $f_{\alpha}$  never has infinitely many periodic points (by the property  $(\star)$ );
- (c) suppose that  $\alpha$  is the right endpoint of some interval  $J$  such that  $\tau(\alpha') = p/q, \alpha' \in J$ . Then the lift of  $f_{\alpha}$  satisfies  $F_{\alpha}^q - \text{Id} - p \geq 0$  and the zeros of  $F_{\alpha}^q - \text{Id} - p$  project to the periodic orbits of  $f_{\alpha}$ . Hence all periodic orbits of  $f_{\alpha}$  are semistable, i.e. attract on one side and repel on the other side. All non-periodic points move in the same direction under iterations of  $f_{\alpha}^q$  (see [6] for more details).

We will inductively choose two sequences of numbers  $\{\alpha_n\}_{n=1}^{\infty}$  and  $\{\tilde{\alpha}_n\}_{n=1}^{\infty}$  satisfying:

- (1)  $\alpha_n, \tilde{\alpha}_n < \delta/2$ ;
- (2)  $\alpha_{n-1} < \tilde{\alpha}_{n-1} < \alpha_n$ ;
- (3)  $\alpha_n = \max \tau^{-1}(p_n/q_n)$ ;
- (4)  $\tau(\tilde{\alpha}_n)$  is Diophantine;
- (5)  $p_n/q_n - p_{n-1}/q_{n-1} < (2(n-1)^2 \max_{1 \leq k \leq n-1} q_k^2)^{-1}$ .



Let  $P_n$  be the number of periodic points of  $f_{\alpha_n}$ . Choose  $r_n > 0$  so small that  $r_n \leq (2^n P_n)^{-n}$  and open intervals of length  $r_n$  centered at the periodic points of  $f_{\alpha_n}$  are disjoint. Let  $I_n$  be the union of these intervals. By (c), there exists a number  $k(\alpha_n, I_n)$  such that the orbit of any point  $x \in S^1$  has at most  $k(\alpha_n, I_n)$  points outside  $I$ .

Any sufficiently small perturbation of  $f_{\alpha_n}$  satisfies the following two properties.

- (i) The time of the first return to  $I_n$  is bounded above by  $k(\alpha_n, I_n) + 1$ .
- (ii) Once a trajectory enters  $I_n$ , the next  $2^{n+2}k(\alpha_n, I_n)$  iterations belong to  $I_n$ .

It follows that there exists  $\varepsilon_n > 0$ , such that for any  $\alpha \in [\alpha_n, \alpha_n + \varepsilon_n)$

$$\frac{1}{N} \sum_{k=0}^N \chi_{I_n}(f_{\alpha}^k(x)) > 1 - \frac{1}{2^n}$$

for any  $N > 2^{n+2}k(\alpha_n, I_n)$  and any  $x \in S^1$ .

We will choose  $\alpha_m, m > n$ , and  $\tilde{\alpha}_l, l \geq n$ , such that:

- (5)  $\alpha_m, \tilde{\alpha}_l \in [\alpha_n, \alpha_n + \varepsilon_n/2)$ .

Now we choose a number  $\tilde{\alpha}_n \in [\alpha_n, \alpha_n + \varepsilon_n/2)$  such that  $\tau(\tilde{\alpha}_n)$  is Diophantine. This implies that  $f_{\tilde{\alpha}_n}$  is smoothly conjugate to the rotation  $R_{\tau(\tilde{\alpha}_n)}$  [5]. Denote by  $\mu_{\tilde{\alpha}_n}$  the unique invariant measure corresponding to  $f_{\tilde{\alpha}_n}$ . Since  $\mu_{\tilde{\alpha}_n}$  has a smooth density there exists  $\tilde{r}_n > 0$  such that for any  $x \in S^1$

$$\frac{\log \mu_{\tilde{\alpha}_n}(B(x, \tilde{r}_n))}{\log \tilde{r}_n} > 1 - \frac{1}{2^{n+1}}$$

There exists  $\tilde{\varepsilon}_n > 0$  such that for any  $\alpha \in [\tilde{\alpha}_n, \tilde{\alpha}_n + \tilde{\varepsilon}_n)$  with irrational  $\tau(\alpha)$ ,

$$\frac{\log \mu_{\alpha}(B(x, \tilde{r}_n))}{\log \tilde{r}_n} > 1 - \frac{1}{2^n}$$

for all  $x \in S^1$ .

We will choose  $\alpha_m, \tilde{\alpha}_m, m > n$ , such that:

- (6)  $\alpha_m, \tilde{\alpha}_m \in [\tilde{\alpha}_n, \tilde{\alpha}_n + \tilde{\varepsilon}_n/2)$ .

Let  $\alpha_* = \lim_{n \rightarrow \infty} \alpha_n$ . Note that the limit exists since the sequence  $\{\alpha_n\}$  is monotone and bounded from above, and  $\alpha_* \leq \delta/2$ .

Now we will show that  $\tau(\alpha_*)$  is indeed irrational. By continuity,  $\tau(\alpha_*) = \lim_{n \rightarrow \infty} p_n/q_n$ . Therefore,

$$\begin{aligned} \tau(\alpha_*) - \frac{p_n}{q_n} &= \sum_{k=n}^{\infty} \left( \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right) \\ &\leq \sum_{k=n}^{\infty} \frac{1}{2k^2 \max_{1 \leq i \leq k} q_i^2} \leq \sum_{k=n}^{\infty} \frac{1}{2k^2 q_n^2} \leq \frac{\pi^2}{12q_n^2} < \frac{1}{q_n^2}. \end{aligned}$$

On the other hand, if  $\tau(\alpha_*) = p/q$  then for  $n \in \mathbb{N}$  such that  $q_n > q$  we have

$$\frac{p}{q} - \frac{p_n}{q_n} = \frac{pq_n - qp_n}{qq_n} > \frac{1}{q_n^2}.$$

Let us denote by  $\mu$  the invariant measure for  $f_{\alpha_*}$ . Since  $\alpha_* \in (\alpha_n, \alpha_n + \varepsilon_n/2]$ ,

$$\frac{1}{N} \sum_{k=0}^N \chi_{I_n}(f_{\alpha_*}^k(x)) > 1 - 2^{-n}$$

for all  $N > 2^{n+2}k(\alpha_n, I_n)$ , and hence  $\mu(I_n) \geq 1 - 2^{-n}$ .

Recall that the set  $I_n$  consists of  $P_n$  open intervals of length  $r_n$ . Let  $\hat{I}_n$  be the union of those intervals  $\Delta_n \subset I_n$  for which  $\mu(\Delta_n) \geq (2^n P_n)^{-1}$ . Since  $r_n \leq (2^n P_n)^{-n}$ , we have

$$\frac{\log \mu(B(x, r_n))}{\log r_n} \leq \frac{1}{n}$$

for any  $x \in \hat{I}_n$ . Therefore, if a point  $x$  belongs to  $\hat{I}_n$  for infinitely many  $n$  then  $\underline{d}_\mu(x) = 0$ . Otherwise,  $x \in \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty (S^1 \setminus \hat{I}_n) = F$ .

Note that  $\mu(\hat{I}_n) \geq \mu(I_n) - P_n(2^n P_n)^{-1} \geq 1 - 2^{-n+1}$ . It follows that  $\mu(\bigcap_{n=m}^\infty (S^1 \setminus \hat{I}_n)) = 0$ , and  $\mu(F) = 0$ . So we conclude that  $\underline{d}_\mu(x) = 0$  for  $\mu$ -a.a.  $x \in S^1$ .

Since  $\alpha_* \in (\tilde{\alpha}_n, \tilde{\alpha}_n + \tilde{\varepsilon}/2]$  for any  $n$ ,  $\bar{d}_\mu(x) \geq 1$  for all  $x \in S^1$ . This together with Lemma 2.1 implies that  $\bar{d}_\mu(x) = 1$  for  $\mu$ -a.a.  $x \in S^1$ . □

Now we will prove Theorem 4.1 using Propositions 4.2 and 4.3. The proof is similar to the proof of Theorem 2.1.

*Proof.* By Proposition 4.2 uniquely ergodic property  $(\star)$  diffeomorphisms are dense in  $\overline{D_I^\omega}$ . Hence using Proposition 4.3 we can construct a dense subset  $Z \subset \overline{D_I^\omega}$  with the following property. For every  $f \in Z$  and  $n > 0$  there exist positive numbers  $\tilde{r}_n < r_n < 2^{-n}$  and a set  $\hat{I}_n$  with  $\mu(\hat{I}_n) > 1 - 2^{-n+1}$  (constructed in the proof of Proposition 4.3) such that

$$\begin{aligned} \frac{\log \mu(B(x, r_n))}{\log r_n} &< \frac{1}{n} \quad \text{for any } x \in \hat{I}_n, \\ \frac{\log \mu(B(x, \tilde{r}_n))}{\log \tilde{r}_n} &> 1 - \frac{1}{n} \quad \text{for any } x \in S^1, \end{aligned}$$

where  $\mu$  is the unique invariant measure for  $f$ .

For any diffeomorphism  $f \in Z$  we can construct a sequence of its neighborhoods,  $\{V_n^f\}_{n=1}^\infty$ , such that for any uniquely ergodic diffeomorphism  $g$  in  $V_n^f$  and its unique invariant measure  $\nu$  we have

$$\begin{aligned} \frac{\log \nu(B(x, r_n))}{\log r_n} &< \frac{2}{n} \quad \text{for any } x \in \hat{I}_n, \\ \frac{\log \nu(B(x, \tilde{r}_n))}{\log \tilde{r}_n} &> 1 - \frac{2}{n} \quad \text{for any } x \in S^1, \end{aligned}$$

and  $\nu(\hat{I}_n) > 1 - 2^{-n+2}$ , where  $\hat{I}_n, r_n$  and  $\tilde{r}_n$  are the same as for  $f$ . Indeed, if  $f$  and  $g$  are sufficiently close in  $C^0$ -topology, their invariant measures are sufficiently close in the weak topology.

Let  $Y_0^\omega = \bigcap_{n=1}^\infty \bigcup_{f \in Z} V_n^f$ . Then  $Y_0^\omega \cap \overline{D_I^\omega}$  and  $Y_0^\omega \cap D_I^\omega$  are residual subsets of  $\overline{D_I^\omega}$  and  $D_I^\omega$ , respectively.

Any uniquely ergodic diffeomorphism  $g$  in  $Y_0^\omega$  satisfies the above condition for some sequence of scales  $\{r_n\}$  and  $\{\tilde{r}_n\}$  which converge to zero. It follows that  $\underline{d}_\nu(x) = 0$ ,  $\bar{d}_\nu(x) = 1$  for  $\nu$ -a.a.  $x \in S^1$ .

Since  $r_n, \tilde{r}_n$  and  $\hat{I}_n$  are as in the proof of Proposition 4.3, we see that the set  $\hat{I}_n$  can be covered by at most  $P_n$  intervals of length  $r_n \leq (2^n P_n)^n$ . Hence  $\log P_n / \log r_n \rightarrow 0$  as  $n \rightarrow \infty$  and we conclude that  $\underline{\dim}_B(\bigcap_{n=k}^\infty \hat{I}_n) = 0$  for any  $k > 0$ . Since  $\nu(\bigcap_{n=k}^\infty \hat{I}_n) > 1 - 2^{-n+3} \rightarrow 1$ , it follows that  $\underline{\dim}_B \nu = 0$ .

On the other hand, since  $\nu(B(x, \tilde{r}_n)) < \tilde{r}_n^{1-2n^{-1}}$  for any  $x \in [0, 1]$ , the minimal number  $N$  of balls of radius needed to cover a set of  $\nu$ -measure  $1 - \varepsilon$  is at least  $(1 - \varepsilon)\tilde{r}_n^{-(1-2n^{-1})}$ . Hence

$$\frac{\log N}{-\log \tilde{r}_n} \geq 1 - \frac{2}{n} + \frac{\log(1 - \varepsilon)}{-\log \tilde{r}_n} \xrightarrow{n \rightarrow \infty} 1$$

and we conclude that  $\overline{\dim}_B \nu \geq 1$ . Since

$$0 \leq \dim_H \nu \leq \underline{\dim}_B \nu \leq \overline{\dim}_B \nu \leq 1$$

for any finite measure on  $S^1$ , we see that

$$\dim_H \nu = \underline{\dim}_B \nu = 0 \quad \text{and} \quad \overline{\dim}_B \nu = 1.$$

This implies that any uniquely ergodic diffeomorphism  $g$  in  $Y_0^\omega$  lies in  $Y^\omega$ . Since the set of the diffeomorphisms in  $\overline{D_I^\omega}$  which are not uniquely ergodic is of the first category, we conclude that  $Y^\omega$  is a residual subset in both  $D_I^\omega$  and  $\overline{D_I^\omega}$ .  $\square$

*Acknowledgements.* We would like to thank Anatole Katok and Yakov Pesin for attracting our attention to this problem and useful discussions. We are grateful to the Pennsylvania State University, where this paper was written. V. Sadovskaya was partially supported by the National Science Foundation grant #DMS9403723.

#### REFERENCES

- [1] D. Anosov and A. Katok. New examples in smooth ergodic theory. Ergodic diffeomorphisms. *Trans. Moscow Math. Soc.* **23** (1970), 1–35.
- [2] L. Barreira, Ya. Pesin and J. Schmeling. Dimension and product structure of hyperbolic measures. *Ann. Math.* **149**(3) (1999), 755–783.
- [3] J. P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors. *Rev. Mod. Phys.* **57** (1985), 617–656.
- [4] K. Falconer. *Fractal Geometry, Mathematical Foundations and Applications*. Wiley, New York, 1990.
- [5] M.-R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Inst. Hautes Études Sci. Publ. Math.* **49** (1979), 5–233.
- [6] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, vol. 54)*. Cambridge University Press, Cambridge, 1995.
- [7] F. Ledrappier and M. Misiurewicz. Dimension of invariant measures for maps with exponent zero. *Ergod. Th. & Dynam. Sys.* **5** (1985), 595–610.
- [8] Ya. Pesin. *Dimension Theory in Dynamical Systems*. University of Chicago Press, Chicago.
- [9] Ya. Pesin and H. Weiss. On the dimension of deterministic and random Cantor-like sets, symbolic dynamics, and the Eckmann–Ruelle conjecture. *Commun. Math. Phys.* **182**(1) (1996), 105–153.