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NEW PROGRESS IN NONUNIFORM MEASURE AND COCYCLE RIGIDITY

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Abstract. We consider an ergodic invariant measure μ for a smooth action α of $\mathbb{Z}^k,\ k \geq 2$, on a $(k+1)$-dimensional manifold or for a locally free smooth action of $\mathbb{R}^k,\ k \geq 2$, on a $(2k+1)$-dimensional manifold. if μ is hyperbolic with the Lyapunov hyperplanes in general position and if one element in $\mathbb{Z}^k$ has positive entropy, then μ is absolutely continuous. The main ingredient is absolute continuity of conditional measures on Lyapunov foliations which holds for a more general class of smooth actions of higher rank abelian groups. We also consider actions on the torus $\mathbb{T}^N$ with induced action on the first homology corresponding to a finite index subgroup of a maximal semisimple abelian subgroup of $SL(N,\mathbb{R})$. Such an action has a unique invariant measure, called large measure, which projects to the Lebesgue measure under the semiconjugacy with the linear action and this measure is absolutely continuous. Finally, we consider cocycles over an action on the torus with Cartan homotopy data. Every cocycle which is Hölder with respect to a Lyapunov Riemannian metric a.e. for the large invariant measure is cohomologous to a constant cocycle via a Lyapunov-Hölder transfer function.

1. Formulation of results

1.1. **Introduction.**

A general program started in [6] and [11] aims at showing that actions of higher rank abelian groups, i.e $\mathbb{Z}^k \times \mathbb{R}^l,\ k+l \geq 2$, by diffeomorphisms of compact manifolds under global conditions of topological or dynamical nature which ensure both infinitesimal hyperbolic behavior and sufficient global complexity of the orbit structure must preserve a geometric structure, such as an absolutely continuous invariant measure.

In [6] and [11] we proved existence of an absolutely continuous measure with strong additional properties for $\mathbb{Z}^k$ actions on the torus $\mathbb{T}^{k+1}$, $k \geq 2$, that induce on $H_1(\mathbb{T}^{k+1},\mathbb{R})$ the action of a maximal rank free abelian subgroup of $SL(k+1,\mathbb{Z})$ diagonalizable over $\mathbb{R}$. We say that such an action has *Cartan homotopy data*.

If the action on the first homology group induced by a $\mathbb{Z}^k$ action α on a torus $\mathbb{T}^N$ contains a hyperbolic element, then there is a semi-conjugacy h between α, and the corresponding linear action α_0 by automorphisms of the torus, i.e. a unique surjective continuous map $h : \mathbb{T}^N \to \mathbb{T}^N$ homotopic to identity such that $h \circ \alpha = \alpha_0 \circ h$. This gives desired global complexity right away and in the Cartan case allows to produce nonuniform hyperbolicity (non-vanishing of the Lyapunov

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exponents) with little effort (see [6, Lemma 2.3]). Existence of the semi-conjugacy allows to use specific properties of the linear action α_0 and reduces the proofs to showing that the semi–conjugacy is absolutely continuous and bijective on an invariant set of positive Lebesque measure.[1] Thus, this may be considered as a version in the setting of global measure rigidity of the *a priori* regularity method developed for the study of local differentiable rigidity in [16] and successfully applied to the global conjugacy problem on the torus in [21].

Definitions of Lyapunov exponents and related notions for $\mathbb{Z}^k$ and $\mathbb{R}^k$ actions by measure preserving diffeomorphisms and their basic properties as well as the suspension construction can be found in [5, Sections 5.1 and 5.2] and [6]. We summarize some essential information in Section 2.

In the present paper we proceed along three different directions.

1.2. **Maximal rank actions on arbitrary manifolds.** Here we make no assumptions on the topology of the ambient manifold or the action under consideration and instead assume directly that the action preserves a measure with non-vanishing Lyapunov exponents whose behavior is similar to that of the exponents for a Cartan action. Namely, we consider a $\mathbb{Z}^k$, $k \geq 2$, action on a $(k+1)$-dimensional manifold or an $\mathbb{R}^k$, $k \geq 2$, action on a $(2k+1)$-dimensional manifold with an ergodic hyperbolic invariant measure for which the kernels of the non-zero Lyapunov exponents called *Lyapunov hyperplanes* are in general position. This means that the dimension of the intersection of any l of those hyperplanes is equal to $k - l$. Dynamical complexity is provided by the assumption that at least one element of the action has positive entropy. Our results for $\mathbb{Z}^k$ actions are direct corollaries of those for $\mathbb{R}^k$ actions via suspension construction.

Theorem 1. *Let μ be an ergodic invariant measure for a $C^{1+\theta}$, $\theta > 0$, action α of $\mathbb{Z}^k$, $k \geq 2$, on a $(k+1)$-dimensional manifold, or for a locally free $C^{1+\theta}$, $\theta > 0$, action α of $\mathbb{R}^k$, $k \geq 2$, on a $2k+1$-dimensional manifold.*

Suppose that the Lyapunov exponents of μ are in general position and that at least one element in $\mathbb{Z}^k$ has positive entropy with respect to μ. Then μ is absolutely continuous.

In Section 3 we present a detailed outline of the proof of Theorem 1 including the key construction of a "uniformizing" time change. For a complete proof see [7].[2]

We develop principal elements of the basic geometric approach of [15] in this general non-uniform setting. This has been done partially already in [6] and we rely on the results and constructions of that paper which do not depend on existence of the semi-conjugacy. The main difference is that the elements within the Lyapunov hyperplanes no longer have the desired derivative estimates for which the semi-conjugacy was used in a critical way. The main difficulty is producing sequences of elements with both the derivative estimates and enough recurrence. One main innovation here is a construction of a particular time change which is smooth along the orbits of the action but only measurable transversally that "straightens out" the expansion and contractions coefficients. This is somewhat similar to the

[1] And in fact smooth in the sense of Whitney on smaller non-invariant sets of positive Lebesgue measure.

[2] We are not aware of any examples of $\mathbb{R}^k$ actions satisfying assumptions of (2) other than time changes of suspensions of $\mathbb{Z}^k$ actions satisfying (1).

"synchronization" time change for Anosov flows introduced by Bill Parry in [20]. The main technical difficulty is showing the new action to possess certain properties as if it were smooth.

Theorem 1 is the first case of existence of an absolutely continuous invariant measure for actions of abelian groups whose orbits have codimension two or higher, which is derived from general purely dynamical assumptions. Nothing of that sort takes places in the classical dynamics for actions of orbit codimension two or higher.[3] Only for codimension one actions (diffeomorphisms of the circle and fixed point free flows on the torus) of sufficient smoothness Diophantine condition on the rotation number (which is of dynamical nature) guarantees existence of a smooth invariant measure [4, 22], and even in those cases the proofs consist of showing that the topological conjugacy by the classical Denjoy theorem is smooth.

1.3. **Actions with linear models with complex eigenvalues.** Aside from the maximal ranks subgroups there are other maximal semisimple (diagonalizable over $\mathbb{C}$) subgroups of $SL(k+1,\mathbb{Z})$. Free elements of such subgroups have j real eigenvalues and l pairs of complex eigenvalues where $j+2l=k$. Let us call an action of $\mathbb{Z}^{j+l}$ which induces on $H_1(\mathbb{T}^{k+1},\mathbb{R})$ such a linear action *an action with maximal homotopy data*. We are able to extend results from [6] and a part of those from [11] to actions with maximal homotopy data.

Let $\mathcal{M}$ be the set of ergodic, α-invariant measures that project to Lebesgue measure λ by the semiconjugacy: $h_*\nu=\lambda$.

Theorem 2. *For any action α of $\mathbb{Z}^{j+l}$, $j+l\geq 2$ on $\mathbb{T}^{j+2l+1}$ by $C^{1+\theta}$, $\theta>0$ diffeomorphisms with maximal homotopy data the set $\mathcal{M}$ consist of a single absolutely continuous measure.*

Detailed outline of the proof of existence of absolutely continuous invariant measure in $\mathcal{M}$ is presented in Section 4. For a complete proof of the theorem see [12].

1.4. **Cocycle rigidity for actions with Cartan homotopy data.** Generally speaking, cocycle rigidity means that one-cocycles of certain regularity over a group action are cohomologous to constant cocycles via transfer functions of certain (often lower) regularity. Cocycle rigidity is prevalent in hyperbolic and partially hyperbolic actions of higher rank abelian groups, see e.g.[10, 13, 14, 3]. Many of those proofs use harmonic analysis and are thus restricted to algebraic actions. The method of [10] based on the TNS (totally non-symplectic) condition (no negatively proportional Lyapunov exponents) can be used in the non-uniformly hyperbolic case and is in particular applicable in the settings considered in the present paper.

One can apply it to smooth or Hölder continuous cocycles but there are reasons to consider two more general classes of cocycles which are in general defined only almost everywhere with respect to a hyperbolic absolutely continuous invariant measure: (i) *Lyapunov Hölder* cocycles: Hölder continuous with respect to a properly defined *Lyapunov metric* which is equivalent to a smooth metric on Pesin sets and changes slowly along the orbits, and (ii)*Lyapunov smooth* cocycles: smooth along invariant foliations at the points of Pesin sets with a similar slow change

[3] In our situation the codimension of orbits is at least three. When codimension of orbits equals two there is not enough space for nontrivial behavior of higher rank actions involving any kind of hyperbolicity, see [9].

condition. Notice that the most important intrinsically defined cocycles, the logarithms of the Jacobians along invariant foliations (Lyapunov, stable, and likewise) are Lyapunov smooth.

Theorem 3. *For any action α of $\mathbb{Z}^k$ on $\mathbb{T}^{k+1}$ with Cartan homotopy data any Lyapunov Hölder (corr. Lyapunov smooth) cocycle is cohomologous to a constant cocycle via a Lyapunov Hölder (corr. Lyapunov smooth) transfer function.*

In Section 5 we define pertinent notions more precisely and present a brief informal outline of the proof of Theorem 3. The proof uses full force of measure rigidity results from [11], specifically Theorem 2.1 and Proposition 2.9. Notice that rigidity of logarithms of Jacobians is already proved in [6]. For a complete proof of Theorem 3 see [12].

In the setting of Theorem 1 we are not able to prove cocycle rigidity (although we conjecture that it should be true) but only a weaker property which, while conceptually interesting, has not found any applications yet.

Proposition 4. *Let μ be a measure from Theorem 1. The spaces of classes of Lyapunov Hölder (corr. Lyapunov smooth) cocycles with respect to cohomology with Lyapunov Hölder (corr. Lyapunov smooth) transfer functions are finite dimensional.*

2. Properties of Lyapunov exponents and invariant manifolds

For a smooth $\mathbb{R}^k$ action α on a manifold M and an element $\mathbf{t} \in \mathbb{R}^k$ we denote the corresponding diffeomorphism of M by $\alpha(\mathbf{t})$. Sometimes we will omit α and write, for example, $\mathbf{t}x$ in place of $\alpha(\mathbf{t})x$ and $D\mathbf{t}$ in place of $D\alpha(\mathbf{t})$ for the derivative of $\alpha(\mathbf{t})x$.

Proposition 2.1. *Let α be a locally free $C^{1+\theta}$, $\theta > 0$, action of $\mathbb{R}^k$ on a manifold M preserving an ergodic invariant measure μ. There are linear functionals χ_i, $i = 1, \dots, l$, on $\mathbb{R}^k$ and an α-invariant measurable splitting, called the* Lyapunov decomposition, *of the tangent bundle of $TM = T\mathcal{O} \oplus \bigoplus_{i=1}^{l} E_i$ over a set of full measure $\mathcal{R}$, where $T\mathcal{O}$ is the distribution tangent to the $\mathbb{R}^k$ orbits, such that for any $\mathbf{t} \in \mathbb{R}^k$ and any nonzero vector $v \in E_i$ the Lyapunov exponent of v is equal to $\chi_i(\mathbf{t})$, i.e.*

$$\lim_{n \to \pm\infty} n^{-1} \log \|D(n\mathbf{t})\, v\| = \chi_i(\mathbf{t}),$$

where $\|\cdot\|$ is any continuous norm on TM. Any point $x \in \mathcal{R}$ is called a regular point.

Furthermore, for any $\varepsilon > 0$ there exist positive measurable functions $C_\varepsilon(x)$ and $K_\varepsilon(x)$ such that for all $x \in \mathcal{R}$, $v \in E_i(x)$, $\mathbf{t} \in \mathbb{R}^k$, and $i = 1, \dots, l$,

1. $C_\varepsilon^{-1}(x) e^{\chi_i(\mathbf{t}) - \frac{1}{2}\varepsilon\|\mathbf{t}\|}\|v\| \le \|D\mathbf{t}\, v\| \le C_\varepsilon(x) e^{\chi_i(\mathbf{t}) + \frac{1}{2}\varepsilon\|\mathbf{t}\|}\|v\|$;
2. *Angles* $\angle(E_i(x), T\mathcal{O}) \ge K_\varepsilon(x)$ *and* $\angle(E_i(x), E_j(x)) \ge K_\varepsilon(x)$, $i \ne j$;
3. $C_\varepsilon(\mathbf{t}x) \le C_\varepsilon(x) e^{\varepsilon\|\mathbf{t}\|}$ *and* $K_\varepsilon(\mathbf{t}x) \ge K_\varepsilon(x) e^{-\varepsilon\|\mathbf{t}\|}$;

The stable and unstable distributions $E^-_{\alpha(\mathbf{t})}$ and $E^+_{\alpha(\mathbf{t})}$ of an element $\alpha(\mathbf{t})$ are defined as the sums of the Lyapunov distributions corresponding to the negative and the positive Lyapunov exponents for $\alpha(\mathbf{t})$ respectively. Stable distributions and hence their transversal intersections are always Hölder continuous (see, for example, [2]).

Proposition 2.2. *Let α be a $C^{1+\theta}$, $\theta > 0$ action of $\mathbb{R}^k$ as in Proposition 2.1. Suppose that a Lyapunov distribution E is the intersection of the stable distributions of some elements of the action. Then E is Hölder continuous on any Pesin set*

$$\mathcal{R}^l_\varepsilon = \{x \in \mathcal{R} : \ C_\varepsilon(x) \le l, K_\varepsilon(x) \ge l^{-1}\} \tag{2.1}$$

with Hölder constant which depends on l and Hölder exponent $\delta > 0$ which depends on the action α only.

Let α be an $\mathbb{R}^k$ action as in Theorem 1. Since $(k+1)$ nontrivial Lyapunov exponents of α with respect to μ are nonzero functionals and the Lyapunov hyperplanes are in general position, the total number of Weyl chambers is equal to $2^{k+1}-2$. Each Weyl chamber corresponds to a different combination of signs for the Lyapunov exponents. In fact, $2^{k+1}-2$ Weyl chambers correspond to all possible combinations of signs except for all pluses and all minuses. The fact that these two combinations are impossible can be seen as follows. First we note that μ is non-atomic since it is ergodic for α and the entropy for some element is positive. Now assume that there is an element $\mathbf{t} \in \mathbb{R}^k$ such that all exponents for $\alpha(\mathbf{t})$ are negative. Then every ergodic component for $\alpha(\mathbf{t})$ is an isolated contracting periodic orbit [9, Proposition 1.3] and hence the measure μ must be atomic. In particular, we obtain the following property. Let χ_i, $i = 1, \dots, k+1$, be the non-zero Lyapunov exponents of the action α and let E_i, $i = 1, \dots, k+1$, be the corresponding Lyapunov distributions.

($\mathcal{C}$) *For every $i \in \{1, \dots, k+1\}$ there exists a Weyl chamber $\mathcal{C}_i$ such that for every $\mathbf{t} \in \mathbb{R}^k \cap \mathcal{C}_i$ the signs of the Lyapunov exponents are*

$$\chi_i(\mathbf{t}) < 0 \text{ and } \chi_j(\mathbf{t}) > 0 \text{ for all } j \neq i.$$

In other words, property ($\mathcal{C}$) implies that each Lyapunov distribution E_i is the full stable distribution for any $\mathbf{t} \in \mathcal{C}_i$.

We will use standard material on invariant manifolds corresponding to the negative and positive Lyapunov exponents (stable and unstable manifolds) for $C^{1+\theta}$ measure preserving diffeomorphisms of compact manifolds, see for example [1, Chapter 4].

We will denote by $\mathcal{W}^-_{\alpha(\mathbf{t})}(x)$ the (global) stable manifold for $\alpha(\mathbf{t})$ at a regular point x. This manifold is an immersed Euclidean space tangent to the stable distribution $E^-_{\alpha(\mathbf{t})}$. The unstable manifold $\mathcal{W}^+_{\alpha(\mathbf{t})}(x)$ is defined as the stable one for $\alpha(-\mathbf{t})$ and thus have similar properties. For an action as in the Theorem 1 property ($\mathcal{C}$) gives that each Lyapunov distribution E coincides with some stable distribution and thus we have the corresponding manifolds $\mathcal{W}(x)$ tangent to E. More generally, these manifolds are defined for any Lyapunov distribution E as in Proposition 2.2. We will call them the leaves of the Lyapunov foliation $\mathcal{W}$. It is customary to use words "distributions" and "foliations" in this setting although these objects are correspondingly measurable families of tangent spaces defined a.e. and measurable families of smooth manifolds which fill a set of full measure.

3. Outline of Proof of Theorem 1

3.1. **Reduction to the Technical Theorem.** The following result is the principal technical tool in our proof of Theorem 1.

Theorem 3.1. *Let μ be a hyperbolic ergodic invariant measure for a locally free $C^{1+\theta}$, $\theta > 0$, action α of $\mathbb{R}^k$, $k \geq 2$, on a compact smooth manifold M. Suppose that a Lyapunov exponent χ is simple and there are no other exponents proportional to χ. Let E be the one-dimensional Lyapunov distribution corresponding to χ.*

Then E is tangent μ-a.e. to a Lyapunov foliation $\mathcal{W}$ and the conditional measures of μ on $\mathcal{W}$ are either atomic a.e. or absolutely continuous a.e.

The assumptions on the Lyapunov exponents in Theorem 3.1 are considerably more general than in the Theorem 1. In particular they may be satisfied for all exponents of a hyperbolic measure for an action of any rank greater than one on a manifold of arbitrary large dimension. As an example one can take restriction of an $\mathbb{R}^k$ action satisfying the assumption of Theorem 1 to any lattice $L \subset \mathbb{Z}^k$ of rank at least two which has trivial intersection with all Lyapunov hyperplanes. For this reason Theorem 3.1 has applications beyond the maximal rank case considered in the Theorem 1. On the other hand, positivity of entropy for some or even all non-zero elements is not sufficient to exclude atomic measures on some of the Lyapunov foliations. Thus application to more general actions may include stronger assumptions on ergodic properties of the measure.

Theorem 1 for $\mathbb{R}^k$ actions is deduced from from the technical Theorem 3.1 as follows.

First we show that existence of an element with positive entropy implies that the conditional measures on every Lyapunov foliation are non-atomic a.e. Applying Theorem 3.1 we obtain that all these measures are absolutely continuous. We conclude the proof by showing that this implies absolute continuity of μ.

We recall that a diffeomorphism has positive entropy with respect to an ergodic invariant measure μ if and only if the conditional measures of μ on its stable and unstable foliations are non-atomic a.e. This follows for example from [19]. Thus if the entropy $h_\mu(\mathbf{t})$ is positive for some element $\mathbf{t} \in \mathbb{R}^k$ then the conditional measures of μ on $\mathcal{W}^+_{\alpha(\mathbf{t})}$ are non-atomic. Then there exists an element $\mathbf{s}$ in a Weyl chamber $\mathcal{C}_i$ such that the one-dimensional distribution $E_i = E^-_{\alpha(\mathbf{s})}$ is not contained in $E^+_{\alpha(\mathbf{t})}$ and thus $E^+_{\alpha(\mathbf{t})} \subset E^+_{\alpha(\mathbf{s})} = \bigoplus_{j \neq i} E_j$. Hence the conditional measures on $\mathcal{W}^+_{\alpha(\mathbf{s})}$ are also non-atomic. This gives $h_\mu(\mathbf{s}) > 0$ which implies that the conditional measures on $\mathcal{W}_i = \mathcal{W}^-_{\alpha(\mathbf{s})}$ must be also non-atomic. Now for any $j \neq i$ consider the codimension one distribution $E'_j = \bigoplus_{k \neq j} E_k = E^+_{\alpha(\mathbf{t}_j)}$ for any $\mathbf{t}_j$ in the Weyl chamber $\mathcal{C}_j$. Since $E_i \subset E'_j$ we see that the conditional measures on the corresponding foliation $\mathcal{W}'_j$ are non-atomic. Hence $h_\mu(\mathbf{t}_j) > 0$ and the conditional measures on $\mathcal{W}_j = \mathcal{W}^-_{\alpha(\mathbf{t}_j)}$ are non-atomic too. We conclude that the conditional measures on every Lyapunov foliation $\mathcal{W}_i$, $i = 1, \ldots, k+1$, are non-atomic and therefore absolutely continuous by Theorem 3.1.

The remaining argument is similar to that in [6]. To prove that μ is absolutely continuous we use the following theorem which is the flow analogue of results in Section 5 of [17] (see [17, Theorem (5.5)] and [19, Corollary H]).

Theorem 3.2. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism with invariant measure μ and assume that $h_\mu(f)$ is equal both to the sum of the positive Lyapunov exponents and to the absolute value of the sum of the negative Lyapunov exponents. If the directions corresponding to zero Lyapunov exponents integrate to a smooth foliation* *and the conditional measures with respect to this central foliation are absolutely continuous, then μ is absolutely continuous with respect to Lebesgue measure.*

Consider $\mathbf{t} \in -\mathcal{C}_i$. Then $\chi_i(\mathbf{t}) > 0$ and $\chi_j(\mathbf{t}) < 0$ for all $j \neq i$. Since the conditional measures on $\mathcal{W}^+_{\alpha(\mathbf{t})}$ are absolutely continuous $h_\mu(\alpha(\mathbf{t})) = \chi_i(\mathbf{t})$ for any $\mathbf{t} \in -\mathcal{C}_i$. By the Ruelle inequality $h_\mu(\alpha(\mathbf{t})) \le -\sum_{j\neq i}\chi_j(\mathbf{t})$ and hence $\sum_{j=1}^{k+1}\chi_j(\mathbf{t}) \le 0$. If the equality holds then Theorem 3.2 gives the absolute continuity. If $\sum_{j=1}^{k+1}\chi_j(\mathbf{t}) < 0$ for all $\mathbf{t}$ in all $-\mathcal{C}_i,\ i = 1,\dots k+1$ then $\bigcup_{i=1}^{k+1}\mathcal{C}_i$ lies in the positive half space of the linear functional $\sum_{j=1}^{k+1}\chi_j$. But this is impossible since there are $\mathbf{t}_i \in \mathcal{C}_i,\ i = 1,\dots k+1$ with $\sum_{i=1}^{k+1}\mathbf{t}_i = 0$.

3.2. **Outline of the proof of Theorem 3.1.** First we show that the Lyapunov distribution E is an intersection of some stable distributions of α. An element $\mathbf{t} \in \mathbb{R}^k$ is called *generic singular* if it belongs to exactly one Lyapunov hyperplane. Let $\mathbf{t} \in L = \operatorname{Ker}\chi$ be a generic singular element. Thus

$$TM = T\mathcal{O} \oplus E^-_{\alpha(\mathbf{t})} \oplus E \oplus E^+_{\alpha(\mathbf{t})}$$

We can take a regular element $\mathbf{s}$ close to $\mathbf{t}$ for which $\chi(\mathbf{s}) > 0$ and all other nontrivial exponents have the same signs as for $\mathbf{t}$. Thus

$$E^-_{\alpha(\mathbf{s})} = E^-_{\alpha(\mathbf{t})} \qquad \text{and} \qquad E^+_{\alpha(\mathbf{s})} = E^+_{\alpha(\mathbf{t})} \oplus E. \tag{3.1}$$

Similarly, we can take a regular element $\mathbf{s}'$ close to $\mathbf{t}$ on the other side of L for which $\chi(\mathbf{s}') < 0$ and $E^+_{\alpha(\mathbf{s}')} = E^+_{\alpha(\mathbf{t})}$ and $E^-_{\alpha(\mathbf{s}')} = E^-_{\alpha(\mathbf{t})} \oplus E$. Therefore,

$$E = E^+_{\alpha(\mathbf{s})} \cap E^-_{\alpha(\mathbf{s}')} = E^-_{\alpha(-\mathbf{s})} \cap E^-_{\alpha(\mathbf{s}')}.$$

We conclude that the Lyapunov distribution E is an intersection of stable distributions and, as in Proposition 2.2, is Hölder continuous on Pesin sets. As in Section 2, E is tangent μ-a.e. to the corresponding Lyapunov foliation $\mathcal{W} = \mathcal{W}^-_{\alpha(-\mathbf{s})} \cap \mathcal{W}^-_{\alpha(\mathbf{s}')}$.

We denote by $\mu^{\mathcal{W}}_x$ the system of conditional measures of μ on $\mathcal{W}$ and by $B^{\mathcal{W}}_r(x)$ the ball in $\mathcal{W}(x)$ of radius r with respect to the induced smooth metric. By ergodicity of μ these conditional measures are either non-atomic of have atoms for μ-a.e. x. Since $\mathcal{W}$ is contracted by some elements of the action, it is easy to see that in the latter case the conditional measures are atomic with a single atom for μ-a.e. x (see, for example, [15, Proposition 4.1]). The main part of the proof is to show that if the conditional measures $\mu^{\mathcal{W}}_x$ are non-atomic for μ-a.e. x, then they are absolutely continuous μ-a.e. To prove this we show that they are Haar with respect to the invariant family of smooth affine parameters on the leaves of $\mathcal{W}$ given by [6, Proposition 3.1, Remark 5], see also Proposition 4.3 below). As in [6], we use existence of a sufficient family of maps on the leaves, which are affine with respect to these parameters and preserve $\mu^{\mathcal{W}}_x$ up to a scalar multiple.

Proposition 3.3. [6, Lemmas 3.9 and 3.10] *For μ- a.e. $x \in \mathcal{R}^l_\varepsilon$ (see (2.1)) and for $\mu^{\mathcal{W}}_x$- a.e. $y \in \mathcal{R}^l_\varepsilon \cap B^{\mathcal{W}}_r(x)$ there exists an affine map $g : \mathcal{W}(x) \to \mathcal{W}(x)$ with $g(x) = y$ which preserves the conditional measure $\mu^{\mathcal{W}}_x$ up to a positive scalar multiple. This implies that for μ - a.e. x the conditional measure $\mu^{\mathcal{W}}_x$ is Haar with respect to the affine parameter on $\mathcal{W}(x)$ and thus is absolutely continuous.*

Such a map g is constructed in [6] as a limit of the restrictions of certain elements of the action to $\mathcal{W}(x)$. To verify the assumptions of Lemmas 3.9 and 3.10 of [6] we establish the following

Proposition 3.4. *For any Pesin set $\mathcal{R}^l_\varepsilon$ there exist positive constants K and l' so that for μ- a.e. $x \in \mathcal{R}^l_\varepsilon$ and for $\mu^{\mathcal{W}}_x$- a.e. $y \in \mathcal{R}^l_\varepsilon \cap B^{\mathcal{W}}_r(x)$ there exists a sequence of elements $\mathbf{t}_j \in \mathbb{R}^k$ with*

(1) $x_j = \alpha(\mathbf{t}_j)x \in \mathcal{R}^{l'}_\varepsilon$,
(2) $x_j \to y$,
(3) $K^{-1} \leq \|D^E_x \alpha(\mathbf{t}_j)\| \leq K$.

In [6] it was possible to choose elements $\mathbf{t}_j$ within the Lyapunov hyperplane $L = \ker\chi$. In our case, to satisfy (3) we have to consider elements outside of L. The main part of the proof is to produce a sequence of such elements with enough recurrence to guarantee (1) and (2). While it is easy to choose *some* elements satisfying (3), they will typically not lie in a subgroup and their recurrence properties are not clear. To overcome this difficulty we construct a special *measurable* time change.

First we define a Lyapunov metric on the Lyapunov distribution E. With some fixed smooth metric $< \cdot, \cdot >$ on M and $\varepsilon > 0$ we define the *Lyapunov metric (or scalar product)* for vectors $u, v \in E(x)$ at a regular point x by

$$< u, v >_{x,\varepsilon} = \int_{\mathbb{R}^k} < (D\mathbf{s})u, (D\mathbf{s})v > \exp(-2\chi(\mathbf{s}) - 2\varepsilon\|\mathbf{s}\|)\, d\mathbf{s} \tag{3.2}$$

We observe using (1) of Proposition 2.1 that for any $\varepsilon > 0$ the integral above converges for any regular point x. The norm generated by this scalar product will be called the *Lyapunov norm* and denoted by $\|\cdot\|_{x,\varepsilon}$. Note that the Lyapunov norm is only measurable, but the original smooth metric gives a uniform below estimate for it, i.e. there exists $C > 0$ such that $\|u\|_{x,\varepsilon} \geq C\|u\|$ for all regular $x \in M$ and all $u \in E(x)$.

Proposition 3.5. *There exist positive constants γ, $K(l, \epsilon)$, and $C(l, \varepsilon)$ such that the ε-Lyapunov metric is Hölder continuous on $\mathcal{R}^l_\varepsilon$ with exponent γ and constant $K(l, \epsilon)$ and satisfies*

$$\|u\|_{x,\varepsilon} \leq C(l, \varepsilon)\|u\|. \tag{3.3}$$

The exponent γ depends only on ε and the action, while $K(l, \epsilon)$ and $C(l, \varepsilon)$ also depend on the Pesin set $\mathcal{R}^l_\varepsilon$.

Denote by D^E_x the restriction of the derivative to E.

Proposition 3.6. *For any regular point x and any $\mathbf{t} \in \mathbb{R}^k$*

$$\exp(\chi(\mathbf{t}) - \varepsilon\|\mathbf{t}\|) \leq \|D^E_x\ \mathbf{t}\|_\varepsilon \leq \exp(\chi(\mathbf{t}) + \varepsilon\|\mathbf{t}\|). \tag{3.4}$$

While convenient, this estimate in itself is insufficient for obtaining (3) of Proposition 3.4. For this we construct a *measurable* time change for which the expansion or contraction in E with respect to this Lyapunov metric is given exactly by the Lyapunov exponent χ. This is the principal new construction of the paper and the proof of the next proposition is one its main technical parts. Let $\mathbf{w} \in \mathbb{R}^k$ be a vector transversal to the Lyapunov hyperplane $L = \ker\chi$.

Proposition 3.7. *For μ-a.e. $x \in M$ and any $\mathbf{t} \in \mathbb{R}^k$ there exists a unique real number $g(x, \mathbf{t})$ such that the function $\mathbf{g}(x, \mathbf{t}) = \mathbf{t} + g(x, \mathbf{t})\mathbf{w}$ satisfies the equality*

$$\|D^E_x \alpha(\mathbf{g}(x, \mathbf{t}))\|_\varepsilon = e^{\chi(\mathbf{t})}. \tag{3.5}$$

The function $\mathbf{g}(x,\mathbf{t})$ *is measurable, Hölder continuous in* x *with exponent* $\gamma > 0$ *on any Pesin set* $\mathcal{R}^l_\varepsilon$*, and* C^1 *close to identity in* $\mathbf{t}$*:*

$$\left\| \frac{\partial \mathbf{g}}{\partial \mathbf{t}}(x,\mathbf{t}) - \mathrm{Id} \right\| \le \varepsilon \quad \text{for a.e. } x \tag{3.6}$$

Proposition 3.8. *The formula* $\beta(\mathbf{t},x) = \alpha(\mathbf{g}(x,\mathbf{t}))x$ *defines a measurable time change of* α*. The new action* β *preserves an invariant measure* ν *on* M *which is equivalent to* μ *with* $\frac{d\nu}{d\mu} = \Delta^{-1}$*, where* $\Delta(x) = \det\left(\frac{\partial \mathbf{g}}{\partial \mathbf{t}}(x,\mathbf{0})\right)$

The definition and uniqueness of $\mathbf{g}(x,\mathbf{t})$ ensures that β is a time change, and (3.6) implies that its determinant $\Delta(x)$ is a measurable function which is L^∞ close to constant 1 on M. Then the existence of the invariant measure ν for β and the formula for its Radon-Nikodim derivative follow from [8].

The time change β is not smooth and $\beta(\mathbf{t},x)$ has no derivative transversally to the orbit. The derivative in (3.5) is that of α for the fixed element $g(x,\mathbf{t}) \in \mathbb{R}^k$. This equation, together with the properties of the Lyapunov norm on Pesin sets, gives the desired estimate (3) of Proposition 3.4 for $\mathbf{t}_j$ of the form $\mathbf{g}(x,\mathbf{t})$ with $\mathbf{t} \in L$. However, to prove recurrence for such elements we use the action of the time change $\beta(\mathbf{t},x)$ with $\mathbf{t}$ in a one-parameter subgroup in L. This is established by the following proposition, which can be applied to both (α,μ) and (β,ν). We say that partition ξ_1 is *coarser* than ξ_2 and write $\xi_1 < \xi_2$ if $\xi_2(x) \subset \xi_1(x)$ for a.e. x.

Proposition 3.9. *For any generic singular element* $\mathbf{t} \in L$ *the partition* $\xi_{\mathbf{t}}$ *into ergodic components of element* $\mathbf{t}$ *is coarser than the measurable hull* $\xi(E)$ *of the foliation corresponding to distribution* E*.*

The proof is given by the following "π-partition trick", which was first introduced in [15] for actions by toral automorphisms and then adapted to the nonuniform setting in [6]. Using a regular element $\mathbf{s}$ as in (3.1) we get

$$\xi_{\mathbf{t}} \le \xi(E^-_{\mathbf{t}}) = \xi(E^-_{\mathbf{s}}) = \pi(\mathbf{s}) = \xi(E^+_{\mathbf{s}}) \le \xi(E).$$

The first inequality follows from the Hopf argument. The fact that the Pinsker algebra $\pi(\mathbf{s})$ coincides with the measurable hulls of both stable and unstable foliations is given by Theorem B in [18]. This explains the proposition for the smooth action (α,μ).

However, we need to use this proposition with (β,ν). This action is not smooth and one has to first make sense of the stable and unstable foliations for β. This is done in the next proposition. We denote by $\mathcal{N}$ the orbit foliation of the one-parameter subgroup $\{t\mathbf{w}\}$.

Proposition 3.10. *For any element* $\mathbf{s} \in \mathbb{R}^k$ *there exists stable "foliation"* $\tilde{\mathcal{W}}^-_{\beta(\mathbf{s})}$ *which is contracted by* $\beta(\mathbf{s})$ *and invariant under the new action* β*. It consists of "leaves"* $\tilde{\mathcal{W}}^-_{\beta(\mathbf{s})}(x)$ *defined for almost every* x*. The "leaf"* $\tilde{\mathcal{W}}^-_{\beta(\mathbf{s})}(x)$ *is a measurable subset of the leaf* $(\mathcal{N} \oplus \mathcal{W}^-_{\alpha(\mathbf{s})})(x)$ *of the form*

$$\tilde{\mathcal{W}}^-_{\beta(\mathbf{s})}(x) = \{\alpha(\varphi_x(y)\mathbf{w})y : y \in \mathcal{W}^-_{\alpha(\mathbf{s})}(x)\},$$

where $\varphi_x : \mathcal{W}^-_{\alpha(\mathbf{s})}(x) \to \mathbb{R}$ *is an almost everywhere defined measurable function. For* x *in a Pesin set the* φ_x *the restriction of* φ_x *to the Pesin set is Hölder continuous with exponent* γ*.*

The proof gives an explicit formula for the function φ_x in terms of the time change so that its graph is contracted by $\beta(\mathbf{s})$. The calculation is similar to finding stable manifolds for a time change of a flow. The corresponding unstable "foliation" $\tilde{\mathcal{W}}^+_{\beta(\mathbf{s})}$ can be obtained as $\tilde{\mathcal{W}}^-_\beta(-\mathbf{s})$. The "foliation" $\tilde{\mathcal{W}}$ corresponding to E is obtained as an intersection of stables. Now all ingredients of Proposition 3.9 are defined. One has to establish equality of the Pinsker algebras for these objects. Extending [18, Theorem B] to this case is of one of the main technical difficulties in our proof.

4. Outline of Existence Proof in Theorem 2

4.1. **Lyapunov exponents.** Let $\mu \in \mathcal{M}$. Denote by $\chi_i, i = 1, \dots, j$ the Lyapunov exponents of the linear action α_0 corresponding to real eigenvalues and by $\chi_i, i = j+1, \dots, j+l$ the Lyapunov exponents of α_0 corresponding to the pairs of complex eigenvalues. Notice that α_0 has the following property similar to the property $(\mathcal{C})$ of linear Cartan actions,

$(\mathfrak{M})$ *For every $i = 1, \dots, j+l$ there is an element $\mathbf{m} \in \mathbb{Z}^k$ such that $\chi_i(\mathbf{m}) < 0$ and $\chi_s(\mathbf{m}) > 0$ for every $s \neq i$.*

Lemma 4.1. *The Lyapunov half-spaces and Weyl chambers for α with respect to μ are the same as those for α_0.*

This does not exclude the possibility of zero exponents. But those may only appear in complex pairs, at most one per pair. Hence, we can order the Lyapunov exponents $\tilde{\chi}$ for (α, μ) in such a way that there are constants

$$c_i > 0,\ i = 1, \dots, j; \quad c_i^+ > c_i^- \geq 0, \qquad i = j+1, \dots, j+l \qquad \text{such that}$$
$$\tilde{\chi}_i = c_i \chi_i \quad \text{for} \quad i = 1, \dots, j$$
$$\tilde{\chi}_i^+ = c_i^+ \chi_i,\ \tilde{\chi}_i^- = c_i^- \chi_i \quad \text{for} \quad i = j+1, \dots, j+l.$$

Corollary 4.2. *For μ a.e. x and every $i = 1, \dots, j$ there is the manifold $\mathcal{W}_i(x)$ associated to $\tilde{\chi}_i$, such that $h\mathcal{W}_i(x) \subset W_i(h(x)) = h(x) + E_i$. For $i = j+1, \dots j+l$, there are three possibilities:*

1. *if $c_i^+ = c_i^-$ then there is the two-dimensional manifold $\mathcal{W}_i(x)$ associated to $\tilde{\chi}_i^\pm$, such that $h(\mathcal{W}_i(x)) \subset W_i(h(x)) = h(x) + E_i$*
2. *if $c_i^+ > c_i^- > 0$ then there is the two dimensional manifold $\mathcal{W}_i(x)$ associated to $\tilde{\chi}_i^\pm$, such that $h(\mathcal{W}_i(x)) \subset W_i(h(x)) = h(x) + E_i$, and also a one dimensional manifold $\mathcal{W}_i^+(x) \subset \mathcal{W}_i(x)$ associated to $\tilde{\chi}_i^+$.*
3. *if $c_i^+ > c_i^- = 0$ then there is only the one-dimensional manifold $\mathcal{W}_i^+(x)$ associated to $\tilde{\chi}_i^+$ such that $h(\mathcal{W}_i(x)) \subset W_i(h(x)) = h(x) + E_i$.*

We will use generic notation $\mathcal{W}$ for all Lyapunov foliations. Eventually we will show that only case (1) takes place.

4.2. **Affine structures.**

Proposition 4.3. *There exists a unique measurable family of $C^{1+\epsilon}$ smooth α-invariant affine parameters on the leaves of $\mathcal{W}(x)$. Moreover, they depend uniformly continuously in $C^{1+\epsilon}$ topology on x within a given Pesin set.*

Our goal is to prove that the semi-conjugacy h is affine as the map from this affine structure to the standard affine structure on the torus. The case of the Lyapunov foliations associated with real eigenvalues, i.e. $i = 1, \dots, j$ is treated the same way as in [6].

Lemma 4.4. *If i is in case 1 or 2 of Corollary 4.2, we have that for a.e. x, $diam(h(W_i(x))) > 0$ and $Leb_2(h(W_i(x))) > 0$. On the other hand, if i is in case (3) of Corollary 4.2, we have that for a.e. x, $diam(h(W_i^+(x))) > 0$.*

Proposition 4.5. *There are sets Λ of measure arbitrary close to one and $K = K(\Lambda) > 0$ such that if $t \in \ker \chi_i$, $x \in \Lambda$, $\alpha(\mathbf{t})(x) \in \Lambda$ and*

(1) *if i as the in case (1) or (2) of Corollary 4.2, then for every unit vector $v \in E_i(x)$*
$$K^{-1} \le \|D\alpha(\mathbf{t})v\| \le K,$$
(2) *if i as in the case (3) of Corollary 4.2, then for $v \in E_i^+(x)$*
$$K^{-1} \le \|D\alpha(\mathbf{t})(v)\| \le K.$$

The proof follows the same general scheme as the proof of Lemma 3.6 in [6]. Proposition 4.5 allows us to prove the intertwining of the affine structures.

Let us showfirst that cases (2) and (3) of Corollary 4.2 cannot occur. Assume the opposite. Following the same proof as in [6] we have that the conditional invariant measures $\mu_{i,x}^+$ are absolutely continuous with respect to Lebesgue on $\mathcal{W}_i^+(x)$, and using this we can prove the following analog of Lemma 4.3. of [6].

Lemma 4.6. *For almost every x, the semiconjugacy intertwines the actions of the group of translations of $\mathcal{W}_i^+(x)$ and the groups of isometries of $W_i(h(x))$. More precisely, for any translation $\tilde{\tau}$ with respect to the affine structure on $\mathcal{W}_i^+(x)$ there is an isometry τ of the plane $W_i(h(x)) = h(x) + E_i$ such that $h \circ \tilde{\tau} = \tau \circ h$.*

This implies impossibility of these two cases because α_0 acts with complex eigenvalues on E_i. Now consider case (1) of Corollary 4.2. In this case we have also recurrence along Weyl chamber walls, that is the π-partition trick applies. Let us denote the conditional measure along $\mathcal{W}_i(x)$ by μ_x^i .

Lemma 4.7. *Given $r > 0$ there are sets Λ of measure arbitrary close to one such that for μ a.e. $x \in \Lambda$ and for μ_x^i-a.e. $y \in \Lambda \cap W_{i,r}(x)$ there exists an affine map $g : \mathcal{W}_i(x) \to \mathcal{W}_i(x)$ with $g(x) = y$ which preserves the conditional measure up to a positive scalar multiple. Furthermore the norms of the derivative of these affine maps are bounded and bounded away of zero uniformly in x and y, for given Λ and r. Moreover, for every such affine map $g : \mathcal{W}_i(x) \to \mathcal{W}_i(x)$, there is an isometry $\tau(g) : h(x) + E_i \to h(x) + E_i$ such that $h \circ g = \tau(g) \circ g$.*

So let us call $G(x)$ the set of all affine maps, $g : \mathcal{W}_i(x) \to \mathcal{W}_i(x)$ such that

(1) g preserves the conditional measure μ_x^i up to a positive scalar multiple,
(2) there is an isometry $\tau(g) : h(x) + E_i \to h(x) + E_i$ such that $h \circ g = \tau(g) \circ g$.

Clearly $G(x)$ is a closed subgroup of affine transformations. Lets as call $A(x) = \{g(x) \text{ s.t. } g \in G(x)\}$. By Lemma 4.7 we have that $\mu_x^i(A(x)^c) = 0$ for a.e. x. On the other hand, by the entropy assumption we have that μ_x^i is not atomic, so we get that $A(x)$ is not discrete. So we have that $\dim A(x)$ is either 1 or 2. In the latter case, we are done, and get that μ_x^i is absolutely continuous and that h is a diffeomorphism for a.e. x. If $\dim A(x) = 1$ we still have that h is a diffeomorphism when restricted to $A(x)$. Let us call $\hat{E}_i(x) = T_x A(x)$. Clearly $\hat{E}_i(x)$ is α-invariant and defined on a set of full μ measure. Then $D_x h \hat{E}_i(x)$ will give us an α_0-invariant one-dimensional sub-bundle of E_i, which is impossible since α_0 does not support such object. Thus μ_x^i is absolutely continuous with respect to Lebesgue for a.e. point. Then the proof ends exactly us in the case of Cartan homotopy data [6].

For reasons of space we do not discuss the uniqueness proof.

5. Outline of proof of Theorem 3

We first need to describe properly the classes of cocycles considered in Theorem 3. Let us fix a small positive number ϵ and consider Pesin sets $\mathcal{R}_\varepsilon^l$ as defined in (2.1). Let us consider Lyapunov Riemannian metric defined on the set of full measure $\mathcal{R}_\varepsilon = \bigcup_l \mathcal{R}_\varepsilon^l$. It is defined similarly to (3.2) with summation over $\mathbb{Z}^k$ instead of integration. By Proposition 3.5 this metric is Hölder continuous on each $\mathcal{R}_\varepsilon^l$. Now consider a system of neighborhoods $P_\varepsilon(x)$ sometimes called *Pesin boxes* of points in $\mathcal{R}_\varepsilon$ whose size depends on l and slowly oscillates with the action, similarly to the function K_ε from Proposition 2.1. Now using a local coordinate system from a fixed finite atlas project the Lyapunov metric from T_x to the Pesin box around x with constant coefficients. Thus we obtain a system of locally defined metrics and cocycle β defined on $\mathcal{R}_\varepsilon$ is called *Lyapunov Hölder* if for any l, $x \in \mathcal{R}_\varepsilon^l$ β is Hölder continuous on $\mathcal{R}_\varepsilon^l \cap P_\varepsilon(x)$ with Hölder exponent and constant independent of x and l. Similarly we define Lyapunov smooth cocycles by requiring smoothness along local stable manifolds of points in $\mathcal{R}_\varepsilon^l$ with uniform bounds on derivative with respect to a Lyapunov metric within Pesin boxes.

Notice that the semi-conjugacy h between α and the linear Cartan action α_0 is bijective on an increasing sequence of compact Pesin sets as well on stable and unstable manifolds of points from those sets with respect to all elements of the action α. The strategy of the proof is to use these bijections to construct cocycles over α_0 and then use the method of [10]. Take the image of a Pesin set $\mathcal{P}$ under the semi-conjugacy. Solution of the coboundary equation along any stable manifold $\mathcal{W}$ of is given by the familiar telescoping sum. By the absolute continuity $\mathcal{W} \cap \mathcal{P}$ has large conditional measure in $\mathcal{W}$ and the union of our Pesin sets has full conditional measure. Now one considers periodic cycles anchored at points of the Pesin sets. Any two successive points in such a cycle lie on a one-dimensional Lyapunov line and any three successive points lie in a stable manifold of some element. One can simply consider the situation after the semi-conjugacy, as a cocycle over the linear action. Arguing as in [10] we deduce that solution can be constructed consistently from a single typical point to the union of Pesin sets which has full measure. Since the semi-connjugacy is bijective on a full measure set and is smooth along almost every stable manifold the solution can be brought back and to be shown Lyapunov Hölder or Lyapunov smooth.

In the absence of semi-conjugacy one can still extend the solution along Lyapunov lines but due to the "holes" in the union of Pesin sets the argument works only locally. This leads to Proposition 4. Even in the absence of such holes the solution can be constructed on the universal cover but cannot in general be projected to the original manifold since a possibility of the action preserving a non-trivial homology class cannot be excluded.

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