On the Classification of Resonance-free Anosov $\mathbb{Z}^k$ Actions

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1. Introduction

In this paper we consider the problem of classification of higher-rank abelian Anosov actions. These are actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$, by commuting diffeomorphisms of a compact manifold with at least one Anosov element. It is conjectured that all irreducible actions of this type are *smoothly* conjugate to algebraic models [10]. We will concentrate on $\mathbb{Z}^k$ actions. In this case, the algebraic models are actions by commuting automorphisms of tori and infranilmanifolds.

We note the difference with the case of Anosov actions of $\mathbb{Z}$, that is, Anosov diffeomorphisms. A well-known conjecture states that they are *topologically* conjugate to automorphisms of tori and infranilmanifolds. The conjugacy is typically only Hölder continuous, so a smooth classification of Anosov diffeomorphisms is impossible. Even the topological classification was obtained only for diffeomorphisms of infranilmanifolds and for diffeomorphisms of codimension 1 [2; 16; 17].

The study of higher-rank abelian actions originated in the work on Zimmer's conjecture that the standard action of $\mathrm{SL}(n, \mathbb{Z})$ on $\mathbb{T}^n$, $n \geq 3$, is locally rigid. This means that any C^1 small perturbation is smoothly conjugate to the original action. In the proof of this conjecture, the smoothness of the conjugacy was established using the action of a diagonalizable subgroup isomorphic to $\mathbb{Z}^{n-1}$ [5; 11]. This motivated Katok and Spatzier to study local rigidity for abelian actions; in [14], they established local rigidity for a broad class of *algebraic* Anosov actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$, $k \geq 2$. The natural next step is to obtain a smooth conjugacy to an algebraic model for actions more general than C^1 small perturbations of these models. This is referred to as *global rigidity* or *classification*.

In [11], Katok and Lewis established the first global rigidity result for certain actions of $\mathbb{Z}^{n-1}$ on $\mathbb{T}^n$. The actions they considered have n transversal one-dimensional foliations that are stable foliations for some Anosov elements. Such actions are called *maximal* Cartan. Recently, this result was extended by Rodriguez Hertz [22] to a broader class of maximal actions on tori assuming the existence of one Anosov element. We note that the case of a torus (or infranilmanifold) is special. In this case, an Anosov element is known to be topologically conjugate to an automorphism, which gives topological conjugacy of the whole action to

an algebraic model. Thus the problem reduces to showing the smoothness of this conjugacy.

For actions on an arbitrary manifold, one does not immediately have a conjugacy to an algebraic model or even an algebraic model itself, which makes this case considerably more difficult. The algebraic structure is constructed using dynamical objects such as coarse Lyapunov distributions, which are the finest nontrivial intersections of stable distributions of various Anosov elements of the action. They naturally extend the notion of stable distributions for Anosov maps.

A coarse Lyapunov distribution can be also be described in terms of Lyapunov exponents (see Section 2.2 for details). For a $\mathbb{Z}$ action generated by an Anosov diffeomorphism, Lyapunov exponents given by the multiplicative ergodic theorem can be viewed as linear functionals on $\mathbb{Z}$. In general, these functionals depend on a given invariant measure, and the corresponding Lyapunov distributions are only measurable. Note, however, that all functionals on $\mathbb{Z}$ are proportional. Moreover, for *any* invariant measure, the sum of all Lyapunov distributions for Lyapunov exponents proportional to a given negative exponent with positive constants of proportionality is precisely the stable distribution of the Anosov diffeomorphism. The Lyapunov exponents for a higher-rank abelian action are linear functionals on $\mathbb{R}^k$ or $\mathbb{Z}^k$. A coarse Lyapunov distribution can be defined as a direct sum of Lyapunov distributions for all Lyapunov exponents positively proportional to a given one. To show that these distributions are continuous and independent of an invariant measure, we must assume the existence of sufficiently many Anosov elements (see Proposition 2.3). This assumption is present in all known classification results when the manifold is not assumed to be a torus.

Another property important in the study of higher-rank abelian actions is TNS. An action is called *totally nonsymplectic* (TNS) if there are no negatively proportional Lyapunov functionals. This is equivalent to the fact that any pair of coarse Lyapunov distributions is contracted by some element of the action. This assumption was introduced in [12; 18] for the study of cocycle rigidity and was also used in measure rigidity results [9; 13]. All discrete actions in global rigidity results [8; 10; 11] are TNS. In this paper the TNS assumption is used primarily to ensure that certain elements act transitively. In particular, it yields sufficient irreducibility of the action. We note that without some irreducibility assumption there is no hope for any type of rigidity, since one can take products of Anosov diffeomorphisms.

The simplest higher-rank actions are those with one-dimensional coarse Lyapunov distributions. Such actions are called *Cartan* and generalize the maximal Cartan actions in [11]. The first classification results on an arbitrary manifold were obtained in [10] for certain Cartan actions of $\mathbb{R}^k$ and $\mathbb{Z}^k$ with $k \geq 3$. Our main result, Theorem 1.1, allows higher-dimensional coarse Lyapunov distributions under the condition of uniform quasiconformality. This means that, for any element b of the action, all vectors in a given coarse Lyapunov distribution at a given point are expanded or contracted by the iterates of b at essentially the same rate (see Section 2.3). This condition is trivially satisfied for Cartan actions. Compared to the

result for $\mathbb{Z}^k$ actions in [10], we also reduce the rank requirement to the optimal $k \geq 2$ and weaken the assumptions on Lyapunov exponents.

THEOREM 1.1.   *Let α be an action of $\mathbb{Z}^k$, $k \geq 2$, by C^∞ diffeomorphisms of a compact connected smooth manifold $\mathcal{M}$. Suppose that*

(1) *all nontrival elements of α are Anosov and at least one is transitive;*
(2) *α is uniformly quasiconformal on each coarse Lyapunov distribution;*
(3) *α is totally nonsymplectic;*
(4) *for any Lyapunov functionals χ_i, χ_j, and χ_l, the functional $(\chi_i - \chi_j)$ is not proportional to χ_l.*

Then a finite cover of α is C^∞ conjugate to a $\mathbb{Z}^k$ action by affine automorphisms of a torus.

We note that, under conditions (1)–(3), the Lyapunov exponents do not depend on the measure (as will follow from Theorem 1.2). Hence it suffices to verify condition (4) for just one invariant measure. This condition is equivalent to

(4′) *for any Lyapunov functionals χ_i, χ_j, and χ_l,*

$$\{a \in \mathbb{R}^k : \chi_i(a) = \chi_j(a)\} \neq \ker \chi_l.$$

This condition is clearly weaker than the "nondegeneracy" assumption in [10] that $\ker \chi_i \cap \ker \chi_j$ is not contained in $\ker \chi_l$. We note that $\mathbb{Z}^k$ actions by automorphisms of nonabelian infranilmanifolds always have resonances of the type $\chi_i = \chi_j + \chi_l$; assumption (4), which is slightly stronger than the absence of such resonances, serves to exclude such actions.

In [8] we obtained rigidity results for similar actions under the assumption that the coarse Lyapunov foliations are pairwise jointly integrable. This assumption can be easily verified for actions on tori. Our approach to the proof of Theorem 1.1 is different from the one in [8]. First we construct a Riemannian metric with respect to which the contraction/expansion is given by the Lyapunov exponents. The following result extends [10, Thm. 1.2] to the case of higher-dimensional coarse Lyapunov distributions.

THEOREM 1.2.   *Let α be a smooth action of $\mathbb{Z}^k$ on a compact connected smooth manifold $\mathcal{M}$ satisfying conditions (1)–(3) of Theorem 1.1. Then the Lyapunov functionals are the same for all invariant measures. Moreover, there exists a Hölder continuous Riemannian metric on $\mathcal{M}$ such that, for any $a \in \mathbb{Z}^k$ and any Lyapunov functional χ,*

$$\|Da(v)\| = e^{\chi(a)}\|v\| \tag{1.1}$$

for any vector v in the corresponding Lyapunov distribution.

After the construction of the metric, our approach diverges from the one in [10], which required rank $k \geq 3$. We prove a special application of the C^r section theorem (Proposition 3.9), which is of independent interest, and then use this result to establish smoothness of coarse Lyapunov distributions. Then we prove smoothness

of the metric constructed in Theorem 1.2 and obtain a smooth conjugacy to an algebraic model.

In Section 2 we introduce the main notions and summarize the results that are used throughout this paper. In Section 3.2 we prove Theorem 1.2, and in Sections 3.3–3.5 we complete the proof of our main result, Theorem 1.1.

2. Preliminaries

Throughout the paper, the smoothness of diffeomorphisms, actions, and manifolds is assumed to be C^∞, even though all definitions and some of the results can be formulated in lower regularity.

2.1. Anosov Actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$

Let a be a diffeomorphism of a compact manifold $\mathcal{M}$. We recall that a is *Anosov* if there exist a continuous a-invariant decomposition of the tangent bundle $T\mathcal{M} = E_a^s \oplus E_a^u$ as well as constants $K > 0$ and $\lambda > 0$ such that, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\|Da^n(v)\| \le Ke^{-\lambda n}\|v\| \quad \text{for all } v \in E_a^s, \\
\|Da^{-n}(v)\| \le Ke^{-\lambda n}\|v\| \quad \text{for all } v \in E_a^u.
\end{aligned}
\tag{2.1}
$$

The distributions E_a^s and E_a^u are called the *stable* and *unstable* distributions of a.

Now we consider a $\mathbb{Z}^k$ action α on a compact manifold $\mathcal{M}$ via diffeomorphisms. The action is called *Anosov* if there is an element that acts as an Anosov diffeomorphism. Throughout this paper, we shall use the same letter for an element of the acting group and for the corresponding diffeomorphism.

For a $\mathbb{Z}^k$ action α there is an associated $\mathbb{R}^k$ action $\tilde{\alpha}$ on a manifold $\widetilde{\mathcal{M}}$ given by the standard suspension construction [7]. We will refer to $\tilde{\alpha}$ as the *suspension* of α. This construction generalizes the suspension flow of a diffeomorphism. Similarly, the manifold $\widetilde{\mathcal{M}}$ is a fibration over the "time" torus $\mathbb{T}^k$ with the fiber $\mathcal{M}$. If α is an Anosov $\mathbb{Z}^k$ action then $\tilde{\alpha}$ is an Anosov $\mathbb{R}^k$ action. An $\mathbb{R}^k$ action is called Anosov if some element $a \in \mathbb{R}^k$ is Anosov in the sense of the following definition.

DEFINITION 2.1. Let α be a smooth action of $\mathbb{R}^k$ on a compact manifold $\mathcal{M}$. An element $a \in \mathbb{R}^k$ is called *Anosov* or *normally hyperbolic* for α if there exist positive constants λ and K and a continuous α-invariant splitting of the tangent bundle

$$
T\mathcal{M} = E_a^s \oplus E_a^u \oplus T\mathcal{O},
$$

where $T\mathcal{O}$ is the tangent distribution of the $\mathbb{R}^k$ orbits and (2.1) holds for all $n \in \mathbb{N}$.

It is well known that, in both the discrete and the continuous case, the distributions E_a^s and E_a^u are Hölder continuous and tangent to the stable and unstable foliations W_a^s and W_a^u, respectively [4]. The leaves of these foliations are C^∞ injectively immersed Euclidean spaces. Locally, the immersions vary continuously

in C^∞ topology. Such foliations are said to have *uniformly C^∞ leaves.* In general, the distributions E^s and E^u are only Hölder continuous transversally to the corresponding foliations.

The set of Anosov elements $\mathcal{A}$ in $\mathbb{R}^k$ is always an open subset of $\mathbb{R}^k$, by the structural stability theorem for normally hyperbolic maps of Hirsch, Pugh, and Shub [4]. An $\mathbb{R}^k$ action is called *totally* Anosov if the set of Anosov elements $\mathcal{A}$ is dense in $\mathbb{R}^k$. Algebraic Anosov actions are always totally Anosov, but this is not known for nonalgebraic actions.

2.2. *Lyapunov Exponents and Coarse Lyapunov Distributions*

We will concentrate on the case of $\mathbb{R}^k$ actions; the case of $\mathbb{Z}^k$ is similar. See [10] and [8] for more details.

Let a be a diffeomorphism of a compact manifold $\mathcal{M}$ preserving an ergodic probability measure μ. By the Oseledec multiplicative ergodic theorem, there exist finitely many numbers χ_i and a measurable splitting of the tangent bundle $T\mathcal{M} = \bigoplus E_i$ on a set of full measure such that the forward and backward Lyapunov exponents of $v \in E_i$ are χ_i. This splitting is called *Lyapunov decomposition.*

Let μ be an ergodic probability measure for an $\mathbb{R}^k$ action α on a compact manifold $\mathcal{M}$. By commutativity, the Lyapunov decompositions for individual elements of $\mathbb{R}^k$ can be refined to a joint invariant splitting for the action. The following proposition from [10] describes the multiplicative ergodic theorem for this case. See [8] for the discrete-time version and [7] for more details on the multiplicative ergodic theorem and related notions for higher-rank abelian actions.

PROPOSITION 2.2. *Let α be a smooth action of $\mathbb{R}^k$ and let μ be an ergodic invariant measure. Then there exist finitely many linear functionals χ on $\mathbb{R}^k$, a set of full measure $\mathcal{P}$, and an α-invariant measurable splitting of the tangent bundle $T\mathcal{M} = \bigoplus E_\chi$ over $\mathcal{P}$ such that, for all $a \in \mathbb{R}^k$ and $v \in E_\chi$, the Lyapunov exponent of v is $\chi(a)$; that is,*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Da^n(v)\| = \chi(a),$$

where $\|\cdot\|$ is a continuous norm on $T\mathcal{M}$.

The splitting $\bigoplus E_\chi$ is called the *Lyapunov decomposition,* and the linear functionals χ are called the *Lyapunov exponents* or *Lyapunov functionals* of α. The hyperplanes $\ker \chi \subset \mathbb{R}^k$ are called the *Lyapunov hyperplanes* or *Weyl chamber walls,* and the connected components of $\mathbb{R}^k - \bigcup_\chi \ker \chi$ are called the *Weyl chambers* of α. The elements in the union of the Lyapunov hyperplanes are called *singular,* and the elements in the union of the Weyl chambers are called *regular.* We note that the corresponding notions for a $\mathbb{Z}^k$ action and for its suspension are directly related; in particular, the nontrivial Lyapunov exponents are the same. Moreover, for the suspension there is one identically zero Lyapunov exponent corresponding to the orbit distribution. From now on, the term Lyapunov exponent will always refer to the nontrivial functionals.

Consider a $\mathbb{Z}^k$ action by *automorphisms* of a torus or an infranilmanifold. In this case, the Lyapunov decomposition is determined by the eigenspaces of the automorphisms, and the Lyapunov exponents are the logarithms of the moduli of the eigenvalues. Hence these exponents are independent of the invariant measure and give uniform estimates of expansion and contraction rates. Also, every Lyapunov distribution is smooth and integrable.

In the nonalgebraic case, the individual Lyapunov distributions are generally only measurable and depend on the given measure. This can be observed already for a single Anosov diffeomorphism. However, the full stable distribution for any measure always agrees with E_a^s. For higher-rank actions, *coarse Lyapunov distributions* play a similar role. For any Lyapunov functional χ, the coarse Lyapunov distribution is the direct sum of all Lyapunov spaces with Lyapunov functionals positively proportional to χ:

$$E^\chi = \bigoplus E_{\chi'}, \quad \chi' = c\chi \text{ with } c > 0.$$

One can see that, for an algebraic action, such a distribution is a finest nontrivial intersection of the stable distributions of certain Anosov elements of the action. For nonalgebraic actions, however, it is not a priori clear that the intersection of several stable distributions has constant dimension and that it is better than measurable. It was shown in [10] that, in the presence of sufficiently many Anosov elements, the coarse Lyapunov distributions are indeed well-behaved dynamical objects. The next proposition summarizes important results for the suspensions of $\mathbb{Z}^k$ actions under consideration (see [10, Prop. 2.4, Lemma 2.5, Cor. 2.8]).

Proposition 2.3. *Let α be a C^∞ action of $\mathbb{Z}^k$ on $\mathcal{M}$ for which all nontrival elements are Anosov and at least one is transitive. Let $\tilde{\alpha}$ be its suspension $\mathbb{R}^k$ action on the manifold $\widetilde{\mathcal{M}}$. Then the following statements hold.*

(1) *There is an $\tilde{\alpha}$-invariant Hölder continuous coarse Lyapunov splitting*

$$T\widetilde{\mathcal{M}} = T\mathcal{O} \oplus \bigoplus E^i, \qquad (*)$$

where $T\mathcal{O}$ is the tangent distribution of the $\mathbb{R}^k$-orbits and E^i are the finest nontrivial intersections of the stable distributions of various Anosov elements of the action. Every distribution E^i is tangent to the Hölder foliation W^i with uniformly C^∞ leaves.

(2) *The Lyapunov hyperplanes and Weyl chambers are the same for all $\tilde{\alpha}$-invariant ergodic probability measures. The set $\mathcal{A}$ of Anosov elements for $\tilde{\alpha}$ is the union of these Weyl chambers in $\mathbb{R}^k$. In particular, $\tilde{\alpha}$ is a totally Anosov $\mathbb{R}^k$ action.*

(3) *For any $\tilde{\alpha}$-invariant ergodic probability measure, the coarse Lyapunov splitting coincides on the set of full measure with the splitting $(*)$. More precisely, for each Lyapunov exponent χ,*

$$E^\chi(p) = \bigcap_{\{a \in \mathcal{A} \,|\, \chi(a) < 0\}} E_a^s(p) = \bigoplus_{\{\chi' = c\chi \,|\, c > 0\}} E_{\chi'}(p).$$

(4) *The action is called totally nonsymplectic, or TNS, if there are no negatively proportional Lyapunov functionals. Under this condition, almost every element of every Lyapunov hyperplane is transitive on $\widetilde{\mathcal{M}}$.*

Property (2) implies that any regular element $a \in \mathbb{R}^k$ is Anosov. It is clear that its stable (resp. unstable) distribution, given by Definition 2.1, is the sum of all coarse Lyapunov distributions with $\chi(a) < 0$ (resp. $\chi(a) > 0$). For a singular element a of $\tilde{\alpha}$, we can define its neutral, stable, and unstable distributions as follows:

$$E_a^0 = T\mathcal{O} \oplus \bigoplus_{\chi(a)=0} E^\chi, \quad E_a^s = \bigoplus_{\chi(a)<0} E^\chi, \quad E_a^u = \bigoplus_{\chi(a)>0} E^\chi.$$

Note that, a priori, we have no uniform estimates on the possible expansion or contraction of E_a^0 by a and so we cannot say that a is a partially hyperbolic element in the usual sense. However, the properties of the action imply the following.

LEMMA 2.4 [10, Lemma 2.6]. *For the action $\tilde{\alpha}$ in Proposition 2.3, the distributions E_a^0, E_a^s, and E_a^u are Hölder continuous. E_a^s and E_a^u integrate to Hölder continuous foliations W_a^s and W_a^u with uniformly C^∞ leaves. E_a^s is uniformly contracted and E_a^u is uniformly expanded by a.*

2.3. Uniform Quasiconformality, Conformality, and Conformal Structures

Let a be a diffeomorphism of a compact Riemannian manifold $\mathcal{M}$, and let E be a continuous a-invariant distribution. The diffeomorphism a is *uniformly quasiconformal* on E if there exists a constant K such that, for all $n \in \mathbb{Z}$ and $x \in \mathcal{M}$,

$$K^E(x, a^n) = \frac{\max\{\|Da^n(v)\| : v \in E(x), \|v\| = 1\}}{\min\{\|Da^n(v)\| : v \in E(x), \|v\| = 1\}} \leq K. \tag{2.2}$$

If $K^E(x, a) = 1$ for all x, the diffeomorphism is said to be *conformal* on E.

We remark that the notion of uniform quasiconformality does not depend on the choice of a Riemannian metric on the manifold. Clearly, an Anosov diffeomorphism can be uniformly quasiconformal on E only if E is contained in its stable or its unstable distribution.

A *conformal structure* on $\mathbb{R}^n$, $n \geq 2$, is a class of proportional inner products. The space C^n of conformal structures on $\mathbb{R}^n$ identifies with the space of real symmetric positive definite $n \times n$ matrices with determinant 1, which is isomorphic to $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$. It is known that the space $C^n = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R})$ carries a $\mathrm{GL}(n, \mathbb{R})$-invariant metric for which C^n is a Riemannian symmetric space of nonpositive curvature (see [24] for more details).

Let E be a distribution on a compact manifold $\mathcal{M}$. For each $x \in \mathcal{M}$, let $C^E(x)$ be the space of conformal structures on $E(x)$. This gives a bundle C^E over $\mathcal{M}$ whose fiber over x is $C^E(x)$. A continuous (smooth, measurable) section of C^E is called a continuous (smooth, measurable) conformal structure on E. A measurable conformal structure τ on E is called *bounded* if the distance between $\tau(x)$ and $\tau_0(x)$ is uniformly bounded on $\mathcal{M}$ for some continuous conformal structure τ_0 on E.

Clearly, a diffeomorphism is conformal with respect to a Riemannian metric on E if and only if it preserves the conformal structure associated with this metric.

Let Γ be a group acting on $\mathcal{M}$ via diffeomorphisms, and let E be a continuous Γ-invariant distribution. We say that the action is uniformly quasiconformal on E if the quasiconformal distortion $K^E(x, \gamma)$ is uniformly bounded for all $x \in \mathcal{M}$ and $\gamma \in \Gamma$. The next proposition easily follows from this definition.

PROPOSITION 2.5 [8, Prop. 2.8]. *Suppose that the Γ-action is generated by finitely many commuting diffeomorphisms. If each generator is uniformly quasiconformal then the Γ-action is uniformly quasiconformal.*

REMARK 2.6. If a $\mathbb{Z}^k$ or an $\mathbb{R}^k$ action is uniformly quasiconformal on a coarse Lyapunov distribution, then there is only one Lyapunov exponent corresponding to this distribution for any given ergodic measure. Therefore, for the actions in Theorem 1.1 and 1.2 there are no positively proportional Lyapunov exponents—in other words, coarse Lyapunov distributions coincide with Lyapunov distributions. Together with the TNS assumption, this also means that there are no proportional Lyapunov exponents.

We note that uniform quasiconformality on a given coarse Lyapunov distribution is stronger than the existence of only one Lyapunov exponent. This can be observed even for algebraic actions with nontrivial Jordan blocks.

3. Proofs

3.1. Outline of the Proof

In Section 3.2 we obtain a Riemannian metric with respect to which the contraction/expansion is given by the Lyapunov exponents. The metric is constructed separately on each coarse Lyapunov distribution E. First we obtain an invariant conformal structure on E. Then we choose a proper normalization using a Livsic-type argument for a transitive element in the Lyapunov hyperplane corresponding to E. This argument is specific to the TNS case and is simpler than the one in [10].

In Section 3.3 we prove a special application of the C^r section theorem of Hirsch, Pugh, and Shub. We apply this result with one Lyapunov foliation as a base and a sum of several Lyapunov distributions as a fiber. This allows us to prove that any Lyapunov distribution is smooth along any Lyapunov foliation and to thereby obtain the smoothness of the Lyapunov splitting.

In Section 3.4 we establish smoothness of the metric constructed in Theorem 1.2. For each coarse Lyapunov distribution, we use holonomy maps to show smoothness of the conformal structure and then use the Livsic theorem to obtain smoothness of the normalization.

In Section 3.5 we show that the metric gives rise to a smooth affine connection that is invariant under the action. This yields smooth conjugacy to an action by automorphisms of an infranilmanifold. We complete the proof by showing that this infranilmanifold is finitely covered by a torus.

3.2. Proof of Theorem 1.2

In this section we prove the following theorem, which gives the corresponding Riemannian metric on the suspension manifold $\widetilde{\mathcal{M}}$. Clearly, such a metric induces the desired metric on the original manifold $\mathcal{M}$ and implies Theorem 1.2.

THEOREM 3.1. *Let α be a smooth action of $\mathbb{Z}^k$ on a compact connected smooth manifold $\mathcal{M}$ satisfying conditions (1)–(3) of Theorem 1.1. Then the Lyapunov exponents are the same for all invariant measures. Moreover, there exists a Hölder continuous Riemannian metric on the suspension manifold $\widetilde{\mathcal{M}}$ such that, for any $a \in \mathbb{R}^k$ and any Lyapunov exponent χ,*

$$\|Da(v)\| = e^{\chi(a)}\|v\| \tag{3.1}$$

for any vector v in the corresponding Lyapunov distribution.

First we observe that the orbit distribution carries a natural metric invariant under the action. This metric is induced by the local diffeomorphism between an orbit and $\mathbb{R}^k$. We can also declare that the coarse Lyapunov distributions and the orbit distribution are pairwise orthogonal. Thus the theorem is reduced to constructing the desired metric separately on each coarse Lyapunov distribution.

For the rest of the proof of the theorem, we fix a coarse Lyapunov distribution E and the corresponding coarse Lyapunov foliation W. We will construct the Riemannian metric on E satisfying condition (3.1).

The first step is the construction of a conformal structure τ on E that is invariant under $\tilde{\alpha}$. The following theorem gives the corresponding structure for the original action α. We note that if E is one-dimensional then this step becomes trivial.

THEOREM 3.2. *Let α be a C^∞ action of $\mathbb{Z}^k$ on a compact connected smooth manifold $\mathcal{M}$ with a transitive Anosov element, and suppose that the action is uniformly quasiconformal on an invariant Hölder continuous distribution E. Then the action preserves a conformal structure τ on E that is Hölder continuous on $\mathcal{M}$. If, in addition, the distribution E is tangential to a foliation W with uniformly C^∞ leaves, then τ is uniformly C^∞ along the leaves of W.*

We say that a function is *uniformly C^r* along the leaves of W if its derivatives up to order r in the directions of W exist and are continuous on the manifold.

Proof of Theorem 3.2. The proof is similar to the proof of [23, Thm. 1.3]. There the corresponding result is obtained for the case of a single Anosov diffeomorphism (or flow) where E is the (strong) stable distribution. We outline the main steps of the proof for $\mathbb{Z}^k$ actions.

First we use an argument by Tukia [24] to obtain an invariant bounded measurable conformal structure on E. We outline the proof for our case.

LEMMA 3.3. *Let Γ be a group acting on a Riemanian manifold $\mathcal{M}$ via diffeomorphisms. Suppose that the action is uniformly quasiconformal on a continuous*

invariant distribution E. Then there exists a Γ-invariant bounded measurable conformal structure $\tilde{\tau}$ on E.

Proof. Let τ_0 be a continuous conformal structure on E. For each point x in $\mathcal{M}$, we consider the set $S(x) = \{\gamma_*^{-1}\tau_0(\gamma x), \gamma \in \Gamma\}$ of pull-backs of conformal structures on the orbit. Since the action is uniformly quasiconformal, it follows that $S(x)$ is a bounded subset of $C^E(x)$, the space of conformal structures on $E(x)$. Because $C^E(x)$ has nonpositive curvature, there exists a unique ball of the smallest radius containing $S(x)$. We denote its center by $\tilde{\tau}(x)$. One can show that the conformal structure $\tilde{\tau}$ is invariant, bounded, and measurable.   □

The next proposition shows that this invariant conformal structure can be made Hölder continuous by changing it on a set of measure 0. Its proof is virtually identical to the proof of [23, Prop. 3.2].

PROPOSITION 3.4.   *Let $\tilde{\tau}$ be a bounded measurable conformal structure on a Hölder continuous distribution E, and suppose that $\tilde{\tau}$ is preserved by a transitive Anosov diffeomorphism a. Then there exists a Hölder continuous a-invariant conformal structure τ on E that coincides with $\tilde{\tau}$ on a set of full Bowen–Margulis measure.*

We recall that the Bowen–Margulis measure μ for a is the unique measure of maximal entropy. Since any other element $b \in \mathbb{Z}^k$ commutes with a, the measure $b_*\mu$ is also a-invariant and has the same entropy as μ. Hence $b_*\mu = \mu$, and μ is invariant under the action. This implies that we can choose a set of full measure that is invariant under the whole action where τ coincides with $\tilde{\tau}$. Since any such set is dense, we conclude that τ is α-invariant.

Now suppose that E is tangential to a foliation W with uniformly C^∞ leaves. As in the proof of [23, Thm. 1.3], one can show the smoothness of the conformal structure τ along the leaves of W as follows. Let b be an element of the action that contracts W. We recall that there exists a family of diffeomorphisms $h_x : W(x) \to E(x)$ that give a nonstationary linearization of b along W (see [23, Prop. 4.1]). One can show that h_x maps the restriction of τ to a leaf $W(x)$ to the conformal structure $\tau(x)$ on $E(x)$ (see [23, Lemma 3.1]) and hence that τ is C^∞ along the leaves of W. This completes the proof of Theorem 3.2.   □

Theorem 3.2 gives an invariant conformal structure for the original $\mathbb{Z}^k$ action α. This conformal structure gives rise to the conformal structure τ on the coarse Lyapunov distribution E for the suspension $\tilde{\alpha}$. We fix a normalization of τ by a smooth positive function to obtain a background metric g_0 on E with respect to which $\tilde{\alpha}$ is conformal. The rest of the proof of Theorem 3.1 consists of constructing a new metric g in the conformal class of g_0 for which condition (3.1) is satisfied.

For any element $b \in \mathbb{R}^k$ we denote by $D_x^E b$ the restriction of its derivative at $x \in \widetilde{\mathcal{M}}$ to $E(x)$. Let $q(x, b)$ be the norm of $D_x^E b$ induced by the metric g_0 on E. Since E and g_0 are Hölder continuous, so is the function $q(x, b)$ for any b. The action is conformal with respect to g_0, so for any $b \in \mathbb{R}^k$ we have

$$q(x, b) = \|D_x^E b\|_{g_0} = \|(D_x b)(v)\|_{g_0} \cdot \|v\|_{g_0}^{-1} \quad \text{for any nonzero } v \in E(x). \quad (3.2)$$

For a positive continuous function ϕ, let g be the new metric E such that $\|v\|_g = \phi(x)\|v\|_{g_0}$ for any $v \in E(x)$. We will write $g = \phi g_0$. Then

$$q(x, b) = \phi(x)\phi(bx)^{-1}\|(D_x b)(v)\|_g \cdot \|v\|_g^{-1},$$

and condition (3.1) is satisfied for an element $b \in \mathbb{R}^k$ with respect to g if and only if

$$\phi(x) \cdot \phi(bx)^{-1} = e^{-\chi(b)} q(x, b) \quad \text{for any } x \in \widetilde{\mathcal{M}}. \quad (3.3)$$

DEFINITION 3.5.   An element $a \in \mathbb{R}^k$ is called *generic singular* if it is contained in exactly one Lyapunov hyperplane.

We take a generic singular element a_0 in the Lyapunov hyperplane $\mathcal{L}$ corresponding to E. We will construct a metric g satisfying condition (3.1) (or, equivalently, (3.3)) for the one-parameter subgroup $\{ta_0\}$. Any element a of this subgroup is neutral on E; that is, $\chi(a) = 0$. Therefore, condition (3.1) simply means that g is preserved by this subgroup. By Proposition 2.3(4), we can choose a_0 to be transitive on $\widetilde{\mathcal{M}}$; hence there exists a point x^* with dense orbit $\mathcal{O}^* = \{(ta_0)x^*\}$. We define a new metric $g^* = \phi g_0$ on E over $\mathcal{O}^*$ as the propagation of g_0 from $E(x^*)$ along this orbit by the derivative of element ta_0. In other words, we choose $\phi((ta_0)x^*) = q(x^*, ta_0)^{-1}$. By the construction, the metric g^* is preserved by the one-parameter subgroup $\{ta_0\}$.

The main part of the proof is to show that g^* is Hölder continuous on $\mathcal{O}^*$. Then it extends to a Hölder continuous Riemannian metric $g = \phi g_0$ on E, which is also preserved by the subgroup $\{ta_0\}$. Now we consider an arbitrary element $b \in \mathbb{R}^k$. By commutativity, the push-forward $b_* g$ of the metric g is again preserved by $\{ta_0\}$. Since b is conformal with respect to $g = \phi g_0$, we observe that $b_* g = \psi g$ for some positive function ψ. Since $\mathcal{O}^*$ is dense, it is easy to see that this function must be constant: $\psi(x) = \psi(x^*) = c$. We conclude that $b_* g = c \cdot g$ on $\widetilde{\mathcal{M}}$; this implies that, for any $x \in \widetilde{\mathcal{M}}$ and any nonzero vector in $E(x)$, the Lyapunov exponent exists and equals $\log c$. This shows that the condition (3.1) is satisfied for any $b \in \mathbb{R}^k$.

To prove that the metric g^* is Hölder continuous on $\mathcal{O}^*$ it suffices to show that, for any point $x \in \mathcal{O}^*$ that returns close to itself under an element $a = ta_0$, the norm $q(x, a)$ is Hölder close to 1. Let $\beta > 0$ be such that all coarse Lyapunov distributions are Hölder continuous with exponent β. We will show that there exist positive constants ε_0 and K such that, if $\text{dist}(x, ax) < \varepsilon_0$ for some $x \in \widetilde{\mathcal{M}}$ and $a = ta_0$ with $t > 1$, then

$$|\log q(x, a)| < K \cdot \text{dist}(x, ax)^{\beta}. \quad (3.4)$$

We will use the following lemma, which can be viewed as a generalization of the Anosov closing lemma to the case of a partially hyperbolic diffeomorphism with integrable neutral distribution. For a point with a close return it gives a nearby point that returns to its leaf of the neutral foliation. Observe that the usual Anosov closing lemma is for a fully hyperbolic system, so that the neutral foliation is trivial in the discrete-time case (Anosov diffeomorphism) and is the orbit foliation

in the continuous-time case (Anosov flow). The main difference in the partially
hyperbolic case is that, unlike the orbit foliation of a hyperbolic flow, the neutral
foliation is not smooth in general. Hence we cannot use the contracting mapping
principle approach, which relies on differentiability of the holonomy of the neutral
foliation. Instead, we use a topological fixed point theorem and deal with stable
and unstable directions separately.

LEMMA 3.6. *Let W be a coarse Lyapunov foliation, let $E = TW$, and let a_0 be a
generic singular element in the corresponding Lyapunov hyperplane. Then there
exist positive constants ε_0, C, and λ such that, for any $x \in \tilde{M}$ and $a = ta_0$ with
$\mathrm{dist}(x, ax) = \varepsilon < \varepsilon_0$, there exist a $y \in \tilde{M}$ and a $\delta \in \mathbb{R}^k$ such that*:

(1) $\mathrm{dist}(x, y) < C\varepsilon$;
(2) $(a + \delta)y \in W(y)$;
(3) $\|\delta\| < C\varepsilon$;
(4) $\mathrm{dist}((sa_0)x, (sa_0)y) < C\varepsilon e^{-\lambda \min\{s, t-s\}}$ *for any* $0 \le s \le t$;
(5) $|\log q(x, a) - \log q(y, a)| < C\varepsilon^{\beta}$, *where $q(x, a)$ is given by* (3.2).

Proof. Note that any element $a = ta_0$ ($t \ne 0$) is also generic singular and that
the neutral, stable, and unstable distributions E_a^0, E_a^s, and E_a^u are the same for all
$t > 0$. By Lemma 2.4, E_a^u (E_a^s) is uniformly expanded (contracted) by a and in-
tegrates to the foliation W_a^u (W_a^s). The distribution E_a^0 is the direct sum $T\mathcal{O} \oplus E$
and hence it integrates to foliation $\mathcal{O}W$. We note that $E_a^u \oplus E$ is also integrable as
the unstable distribution of an Anosov element close to a. Therefore, $E_a^u \oplus E \oplus \mathcal{O}$
also integrates to the foliation, which we denote by F^u.

First we find a point z close to x such that $az \in F^u(z)$. We consider the ho-
lonomy map of foliation F^u between the stable leaves $W_a^s(x)$ and $W_a^s(ax)$. Let
$B_r^s(x)$ denote the ball of radius r in $W_a^s(x)$. If ε_0 is chosen sufficiently small,
then the balls $B_1^s(x)$ and $B_1^s(ax)$ are close and the holonomy map $H : B_1^s(ax) \to
W_a^s(x)$ along the leaves of F^u is well-defined. Moreover, since the angles between
the tangent spaces to W^s and F^u are uniformly bounded away from 0, there exists
a constant C_1 such that $\mathrm{dist}(x, H(ax)) \le C_1\varepsilon$ and H is close to an isometry:

$$|\mathrm{dist}(H(z'), H(z'')) - \mathrm{dist}(z', z'')| \le C_1\varepsilon \quad \text{for all} \ \ z', z'' \in B_1^s(ax).$$

We can also assume that the time t of the ε-return must be large enough for a to
contract W_a^s by a factor of at least 2. Now it is easy to see that for $B = B_{4C_1}^s$ we
have $H(a(B)) \subset B$. Indeed, for any $z \in B$,

$$\mathrm{dist}(x, H(az)) \le \mathrm{dist}(x, H(ax)) + \mathrm{dist}(H(ax), H(az))$$
$$\le C_1\varepsilon + \mathrm{dist}(ax, az) + C_1\varepsilon \le C_1\varepsilon + \tfrac{1}{2}\mathrm{dist}(x, z) + C_1\varepsilon \le 4C_1.$$

Hence, by the Brouwer fixed point theorem, there exists a point $z \in B$ such that
$H(a(z)) = z$, that is, $az \in F^u(z)$.

Now we can apply a similar argument to find the desired point y. Since the
leaves of $\mathcal{O}W$ foliate the leaves of $F^u(z)$, we can consider the holonomy map
H of foliation $\mathcal{O}W$ from $W_a^u(z)$ to $W_a^u(az)$ inside the leaf $F^u(z)$. As before, we

obtain a fixed point for the map $H_1 \circ a^{-1} \colon B \subset W_a^u(az) \to W_a^u(z)$. This gives the existence of the point $y \in W_a^u(z)$ for which $ay \in \mathcal{O}W(y)$ with $\mathrm{dist}(z, y) \le C_2\varepsilon$. Hence there exists a $\delta \in \mathbb{R}^k$ with $\|\delta\| < C_3\varepsilon$ such that condition (2) is satisfied.

Since by construction we have $z \in W_a^s(x)$ and $y \in W_a^u(z)$, one can easily see that (4) is satisfied. Then (5) follows from the standard estimate for a Hölder expansion/contraction coefficient along exponentially close trajectories. $\square$

We will now complete the proof of the theorem by establishing (3.4). We use the notation of the previous lemma and let $b = a + \delta$. Since δ is small we have $|\log q(y, a) - \log q(y, b)| < C_1\varepsilon$. Together with Lemma 3.6(5), this implies

$$|\log q(x, a) - \log q(y, b)| < C_2\varepsilon^\beta, \tag{3.5}$$

which reduces the proof of (3.4) to showing that

$$|\log q(y, b)| < C_3\varepsilon^\beta. \tag{3.6}$$

In order to show this, we take an Anosov element c that contracts W. Let $y_* = \lim(t_n c)y$ be an accumulation point of the c-orbit of y. We observe that y_* is a fixed point for b. Indeed, using commutativity and the fact that $by \in W(y) \subset W_c^s(y)$, we obtain $by_* = \lim b((t_n c)y) = \lim(t_n c)(by) = y_*$. Our goal is to show that $q(y_*, b)$ is close to 1 and then to show that $q(y_*, b)$ is close to $q(y, b)$, thereby obtaining (3.6).

Because δ is small, we may assume that b is not contained in any Lyapunov hyperplane except possibly for the Lyapunov hyperplane $\mathcal{L}$ corresponding to W. If $b \in \mathcal{L}$, then for the fixed point y_* we must have $q(y_*, b) = 1$. Indeed, $q(y_*, b) \neq 1$ is impossible, since the Anosov elements close to b on one side of $\mathcal{L}$ contract E while those on the other side of $\mathcal{L}$ expand E. If $b \notin \mathcal{L}$ then b is regular and hence Anosov. Then it is well known that the orbit $\mathbb{R}^k y_*$ of its fixed point y_* must be compact [21]. The Lyapunov exponents and the Lyapunov splitting are defined everywhere on this compact orbit. Let $\tilde{\chi}$ be the Lyapunov exponent on this orbit corresponding to E. By Proposition 2.3 we have $\ker \tilde{\chi} = \mathcal{L}$. Hence $\tilde{\chi}(a) = 0$ and we obtain

$$|\tilde{\chi}(b)| = |\tilde{\chi}(a) + \tilde{\chi}(\delta)| = |\tilde{\chi}(\delta)| < C_4\|\delta\| < C_5\varepsilon.$$

We note that $\tilde{\chi}(b) = \log(q(y_*, b))$, since y_* is a fixed point for b. Thus we conclude that in both cases, $b \in \mathcal{L}$ and $b \notin \mathcal{L}$,

$$|\log q(y_*, b)| < C_5\varepsilon. \tag{3.7}$$

Now we will show that

$$|\log q(y, b) - \log q(y_*, b)| \le C_6\varepsilon^\beta. \tag{3.8}$$

Using commutativity, we can write $b = (-t_n c) \circ b \circ (t_n c)$ and obtain

$$q(y, b) = q(b(t_n c)y, -t_n c) \cdot q((t_n c)y, b) \cdot q(y, t_n c).$$

Since $(t_n c)y \to y_*$, the middle term $q((t_n c)y, b)$ tends to $q(y_*, b)$. On the other hand, the product of the other two terms is close to 1:

$$|\log[q(y,t_nc)\cdot q(b(t_nc)y,-t_nc)]| = |\log q(y,t_nc) - \log q(by,t_nc)| < C_6\varepsilon^\beta.$$

For the last inequality we note that c-orbits of y and by are exponentially close and hence the contraction coefficients $q(by,t_nc)$ and $q(y,t_nc)$ are Hölder close by the standard telescopic product argument. This completes the proof of (3.8).

The inequalities (3.8) and (3.7) yield (3.6), which together with (3.5) gives the desired Hölder estimate (3.4). This completes the proof of Theorem 1.2.

3.3. C^r Section Theorem and Smoothness of Coarse Lyapunov Splitting

In this section we prove that, under the assumptions of Theorem 1.1, all coarse Lyapunov distributions are C^∞. We will use the following, a special version of the C^r section theorem (see Theorems 3.1, 3.2, and 3.5 and Remarks 1 and 2 after Theorem 3.8 in [4]).

THEOREM 3.7 [4]. *Let f be a C^r $(r \geq 1)$ diffeomorphism of a compact smooth manifold $\mathcal{M}$, and let W be an f-invariant topological foliation with uniformly C^r leaves. Let $\mathcal{B}$ be a normed vector bundle over $\mathcal{M}$ and let $F : \mathcal{B} \to \mathcal{B}$ be a linear extension of f such that both $\mathcal{B}$ and F are uniformly C^r along the leaves of W.*

Suppose that F contracts fibers of $\mathcal{B}$. That is, for any $x \in \mathcal{M}$ and any $u \in \mathcal{B}(x)$, suppose

$$\|F(u)\|_{fx} \leq k_x\|u\|_x \quad \text{with} \quad \sup_{x\in\mathcal{M}} k_x < 1. \tag{3.9}$$

Then there exists a unique continuous F-invariant section of $\mathcal{B}$. Moreover, if

$$\sup_{x\in\mathcal{M}} k_x\alpha_x^r < 1 \quad \text{where} \quad \alpha_x = \|(df|_{TW(x)})^{-1}\|, \tag{3.10}$$

then the unique invariant section is uniformly C^r smooth along the leaves of W.

REMARK 3.8. If $\alpha_x \leq 1$ for all x (i.e., if f does not contract W), then (3.10) follows from (3.9) and the invariant section is uniformly C^r along the leaves of W.

In Theorem 3.7 we consider the smoothness of the invariant section only along the leaves of the invariant foliation W, so we need the smoothness of the bundle $\mathcal{B}$ and the extension F only along W. One can easily see from the proof in [4] that the contraction in the base (3.10) needs to be estimated only along W. One can also obtain this by formally applying the C^r section theorem with the base manifold $\mathcal{M}$ considered as the disjoint union of the leaves of W. This has been observed in the study of partially hyperbolic systems (see the Introduction, Theorem 3.1, and the subsequent remarks in [20]).

We use Theorem 3.7 to obtain the following corollary on smoothness of invariant distributions. In contrast to the usual regularity results for Anosov and partially hyperbolic systems, the distribution E here is not $T\mathcal{M}$ and is not contained in TW. This gives us greater flexibility, which we will use to show the smoothness of coarse Lyapunov distributions.

PROPOSITION 3.9.   *Let f be a diffeomorphism of a compact smooth manifold $\mathcal{M}$. Let W be an f-invariant topological foliation with uniformly C^∞ leaves such that $\|(Df|_{TW(x)})^{-1}\| \leq 1$ for all $x \in \mathcal{M}$. Let E_1 and E_2 be continuous f-invariant distributions on $\mathcal{M}$ such that the distribution $E = E_1 \oplus E_2$ is uniformly C^∞ along the leaves of W and such that, for any x in $\mathcal{M}$,*

$$\max\{\|Df(v)\| : v \in E_2(x), \|v\| = 1\} < \min\{\|Df(v)\| : v \in E_1(x), \|v\| = 1\}.$$

Then E_1 is uniformly C^∞ along the leaves of W.

Proof. We apply the C^r section theorem (Theorem 3.7) and Remark 3.8. The distribution E is a continuous bundle over $\mathcal{M}$ that is uniformly C^∞ along the leaves of W. There exist distributions $\bar{E}_1$ and $\bar{E}_2$ in E that are close to E_1 and E_2 (respectively) and C^∞ along the leaves of W. Now we can consider a vector bundle $\mathcal{B}$ whose fiber over x is the set of linear operators from $\bar{E}_1(x)$ to $\bar{E}_2(x)$ with the standard operator norm. The differential of f induces a natural action F on $\mathcal{B}$.

The assumptions on f imply that $\sup_{x \in M} k_x < 1$—in other words, that the induced map F contracts the fibers. Indeed, distributions $\bar{E}_1$ and $\bar{E}_2$ can be chosen sufficiently close to E_1 and E_2 such that, for all x in $\mathcal{M}$,

$$\max\{\|Df(v)\| : v \in \bar{E}_2(x), \|v\| = 1\} < \min\{\|Df(v)\| : v \in \bar{E}_1(x), \|v\| = 1\}.$$

Then for all x in $\mathcal{M}$ we can take

$$k_x = \frac{\max\{\|Df(v)\| : v \in \bar{E}_2(x), \|v\| = 1\}}{\min\{\|Df(v)\| : v \in \bar{E}_1(x), \|v\| = 1\}} < 1.$$

Since k_x depends continuously on x, we have $\sup_{x \in M} k_x < 1$. Hence there exists a continuous F-invariant section and, by uniqueness, the graphs of this section give the distribution E_1. Moreover, since $\alpha_x = \|(df|_{TW(x)})^{-1}\| \leq 1$, it follows that the distribution E^1 is C^∞ smooth along the leaves of W. $\square$

Now we use Proposition 3.9 to show the smoothness of the coarse Lyapunov distributions. First we fix a coarse Lyapunov foliation W and show that any coarse Lyapunov distribution is smooth along its leaves.

PROPOSITION 3.10.   *Let W be a coarse Lyapunov foliation. Under the assumptions of Theorem 1.1, any coarse Lyapunov distribution is uniformly C^∞ along the leaves of W.*

Proof. We consider a generic two-dimensional subspace π in $\mathbb{R}^k$ that intersects the Lyapunov hyperplanes along distinct lines. For any Lyapunov exponent χ_j, we denote by $\mathcal{H}_j$ the half-plane in π on which χ_j is negative. We now order these half-planes and the corresponding Lyapunov exponents counterclockwise so that W is the Lyapunov foliation W^1 corresponding to χ_1. Then there exists a unique $i > 1$ such that $\mathcal{H}_1 \cap \mathcal{H}_j$ is nonempty for $1 \leq j \leq i$ and $(-\mathcal{H}_1) \cap \mathcal{H}_j$ is nonempty for $j > i$. Note that, by the TNS assumption (3), $-\mathcal{H}_1 \neq \mathcal{H}_j$ for any j.

We observe that $\bigoplus_{j=1}^{i} E^j = E_a^s$ for any Anosov element a in $\mathcal{H}_1 \cap \mathcal{H}_i$. As a result, this distribution is uniformly C^∞ along W_a^s and in particular along W^1. We

will apply Proposition 3.9 with $W = W^1$, $E = \bigoplus_{j=1}^{i} E^j$, and various choices for E_1, E_2, and f.

Recall that assumption (4) of the theorem is equivalent to (4′) and implies that the hyperplanes $\{a \in \mathbb{R}^k : \chi_i(a) = \chi_j(a)\}$ and $\ker \chi_1$ do not coincide for distinct i and j. Therefore, the plane π can be chosen in such a way that $\chi_i(b) \neq \chi_j(b)$ for any element $b \neq 0$ in $\ker \chi_1 \cap \pi$ and any $i \neq j$. We take an element $b \in \ker \chi_1 \cap \pi$ such that $b \in \mathcal{H}_j$ for $1 < j \leq i$. Since the values $\chi_j(b)$ $(1 \leq j \leq i)$ are all different, we can reorder the indexes $1, \ldots, i$ so that $\chi_{j_i}(b) < \cdots < \chi_{j_2}(b) < \chi_{j_1}(b) = 0$, where $j_1 = 1$.

We fix m $(1 \leq m < i)$ and apply Proposition 3.9 with $f = b$, $E_1 = \bigoplus_{l=1}^{m} E^{j_l}$, and $E_2 = \bigoplus_{l=m+1}^{i} E^{j_l}$. Note that

$$\|Db(v)\| \leq e^{\chi_{m+1}(b)}\|v\| \quad \text{for any } v \in E_2,$$

$$\|Db(v)\| \geq e^{\chi_m(b)}\|v\| \quad \text{for any } v \in E_1$$

with respect to the metric given by Theorem 1.2. Since $\chi_{m+1}(b) < \chi_m(b)$, Proposition 3.9 implies that $\bigoplus_{l=1}^{m} E^{j_l}$ is uniformly C^∞ along the leaves of W^1.

On the other hand, if we take $E_1 = \bigoplus_{l=m+1}^{i} E^{j_l}$, $E_2 = \bigoplus_{l=1}^{m} E^{j_l}$, and $f = b^{-1}$, we obtain that $\bigoplus_{l=m+1}^{i} E^{j_l}$ is uniformly C^∞ along W^1. We conclude that, for any m $(1 \leq m < i)$, both distributions $\bigoplus_{l=1}^{m} E^{j_l}$ and $\bigoplus_{l=m+1}^{i} E^{j_l}$ are uniformly C^∞ along W^1. Since any coarse Lyapunov distribution E^j $(1 \leq j \leq i)$ can be obtained as an intersection of such distributions, it is also uniformly C^∞ along the leaves of W^1.

The smoothness of the distributions E^j $(j > i)$ along W^1 can be obtained similarly. □

COROLLARY 3.11. *Under the assumptions of Theorem 1.1, any coarse Lyapunov distribution is C^∞ on $\mathcal{M}$. In particular, the stable and unstable distributions of any element of the action are C^∞.*

Proof. Proposition 3.10 implies that any coarse Lyapunov distribution E is uniformly C^∞ along the leaves of any coarse Lyapunov foliation. To conclude that it is C^∞ on $\mathcal{M}$, we apply the following lemma with E considered as a function from $\mathcal{M}$ to the Grassmann bundle over $\mathcal{M}$ of subspaces of dimension dim E. □

LEMMA 3.12. *Let ϕ be a map from $\mathcal{M}$ to a finite-dimensional C^∞ manifold. If ϕ is uniformly C^∞ along the leaves of every coarse Lyapunov foliation, then it is C^∞ on $\mathcal{M}$.*

This lemma is a well-known result (see e.g. [3; 11]) obtained by inductive application of the Journé lemma [6].

3.4. Smoothness of the Metric Constructed in Theorem 1.2

PROPOSITION 3.13. *Under the assumptions of Theorem1.1, the metric g constructed in Theorem 1.2 is C^∞.*

REMARK 3.14.   Because we have already obtained the smoothness of the coarse Lyapunov splitting, we do not use assumption (4) in the proof of this proposition.

Proof of Proposition 3.13. Recall that we constructed the metric on each coarse Lyapunov distribution separately and then declared these distributions pairwise orthogonal. Since all coarse Lyapunov distributions are C^∞, it suffices to show that the metric on each coarse Lyapunov distribution is C^∞ on $\mathcal{M}$. First we will establish the smoothness of the conformal structure τ obtained in Theorem 3.2 and then the smoothness of the normalization function ϕ.

We fix a coarse Lyapunov distribution E and the corresponding coarse Lyapunov foliation W. By Theorem 3.2, the conformal structure τ on E is uniformly C^∞ along the leaves of W. We will now show that τ is C^∞ on $\mathcal{M}$.

We consider Anosov elements $a, b \in \mathbb{Z}^k$ such that $\chi(a) < 0 < \chi(b)$ and such that a and b are not separated by any other Lyapunov hyperplane. Recall that, under the assumptions of the theorem, no other Lyapunov exponent has the same kernel (see Remark 2.6). Thus, for any other Lyapunov exponent χ', it follows that $\chi'(a)$ and $\chi'(b)$ have the same sign. It is easy to see that

$$E = E_a^s \cap E_b^u, \quad E \oplus E_b^s = E_a^s, \quad E \oplus E_a^u = E_b^u.$$

We will show that τ is uniformly C^∞ along the leaves of W_b^s. Note that W_b^s is a subfoliation of W_a^s and that both foliations are C^∞ by Corollary 3.11. We consider two nearby points x, y on the same leaf W_b^s and the holonomy map $H_{x,y} \colon W(x) \to W(y)$ given by foliation W_b^s within the leaf of W_a^s. We will show that the map $H_{x,y}$ is conformal (i.e., that it preserves the conformal structure τ) by giving an argument similar to the one in the proof of [23, Thm. 1.4]. We can write

$$H_{x,y} = a^{-n} \circ H_{a^n x, a^n y} \circ a^n,$$

where $H_{a^n x, a^n y}$ is the corresponding holonomy from $W(a^n x)$ to $W(a^n y)$. Since $\text{dist}(a^n x, a^n y) \to 0$ as $n \to \infty$, the derivative $D_{a^n x} H_{a^n x, a^n y}$ of $H_{a^n x, a^n y}$ at $a^n x$ becomes close to an isometry. Since a is conformal, we conclude that the derivative of $a^{-n} \circ H_{a^n x, a^n y} \circ a^n$ at x becomes close to conformal. Taking the limit, we obtain that the holonomy $H_{x,y}$ is conformal. The foliation W_b^s is smooth and τ is preserved by the corresponding holonomy maps, so we conclude that τ is uniformly C^∞ along the leaves of W_b^s.

The smoothness along the leaves of W_a^u is obtained similarly. Since the conformal structure τ is uniformly C^∞ along the leaves of W_b^s, W_a^u, and W, Lemma 3.12 yields the smoothness of τ on $\mathcal{M}$.

To complete the proof of the proposition, it remains to show that the renormalization function ϕ from the proof of Theorem 1.2 is C^∞ on $\mathcal{M}$. We recall that, for any $b \in \mathbb{Z}^k$ and $x \in \mathcal{M}$, the function ϕ satisfies the cohomological equation

$$\phi(x) \cdot \phi(bx)^{-1} = e^{-\chi(b)} q(x, b)$$

with $q(x, b) = \|Db|_{E(x)}\|_{g_0}$, where the metric g_0 is a normalization of τ. Since both g_0 and E are C^∞, the function $q(x, b)$ is C^∞ on $\mathcal{M}$. Now it follows from the Livsic theorem (see [15]) that the continuous solution ϕ is, in fact, C^∞.   □

3.5. Smooth Conjugacy to a Toral Action

The smoothness of the coarse Lyapunov splitting and the existence of a smooth metric g satisfying (1.1) give a strong indication of rigidity. If all coarse Lyapunov distributions are one-dimensional then the algebraic structure can be easily obtained, since the vector fields of unit length in coarse Lyapunov directions span a finite-dimensional Lie algebra [10]. In our case it can be shown that the metric g is flat, but this requires considerable effort. Instead, we use the metric g to obtain an α-invariant C^∞ affine connection on $\mathcal{M}$ and complete the proof similarly as in [8].

For each coarse Lyapunov foliation W^i, we denote by ∇^i the Levi–Civita connection of the induced Riemannian metric on the leaves of W^i. This connection is α-invariant because any element of the action is a homothety on the leaves of W^i. Since both the metric and W^i are C^∞ on $\mathcal{M}$, so is the connection. Now we define an affine connection ∇ on $\mathcal{M}$ using a standard construction. Let X and Y be two vector fields on $\mathcal{M}$. We use the coarse Lyapunov splitting $T\mathcal{M} = \bigoplus E^i$ to decompose $X = \sum X^i$ and $Y = \sum Y^i$, where $X^i, Y^i \in E^i$. Then

$$\nabla_X Y = \sum_i \nabla^i_{X^i} Y^i + \sum_{i \neq j} \Pi_j [X^i, Y^j],$$

where Π_j is the projection onto E^j and defines an affine connection. Since the distributions E^i and the connections ∇^i are C^∞, it follows that ∇ is also C^∞. Since ∇^i and E^i are α-invariant, so is ∇.

Now we consider a transitive Anosov element of α. This element preserves the C^∞ affine connection ∇ and has C^∞ stable and unstable distributions by Corollary 3.11. It is well known (see e.g. [1]) that such a diffeomorphism is conjugate to an Anosov automorphism of an infranilmanifold $\mathcal{N}$ by a C^∞ diffeomorphism h. Then h conjugates the whole action α to an action $\bar\alpha$ by affine automorphisms of $\mathcal{N}$. This follows from the fact that any diffeomorphism commuting with an Anosov automorphism of an infranilmanifold is an affine automorphism itself (see [5, proof of Prop. 2.18] and [19, proof of Prop. 0]).

It remains to show that the infranilmanifold $\mathcal{N}$ is finitely covered by a torus. Recall that $\mathcal{N}$ is finitely covered by a nilmanifold N/Γ, where N is a simply connected nilpotent Lie group and Γ is a cocompact lattice in N. We need to show that N is abelian. The Lie algebra $\mathfrak{n}$ of N splits into Lyapunov subspaces $\mathfrak{e}_i$ with Lyapunov functionals χ_i of the action $\bar\alpha$. We note that since the conjugacy h is smooth, the Lyapunov functionals of the actions α and $\bar\alpha$ coincide. In particular, $\bar\alpha$ has no proportional Lyapunov exponents and no resonances of the type $\chi_l = \chi_i + \chi_j$ among the Lyapunov exponents.

If for $u \in \mathfrak{e}_i$ and $v \in \mathfrak{e}_j$ the bracket $[u, v] \neq 0$, then $[u, v]$ belongs to a nontrivial Lyapunov subspace with Lyapunov functional $\chi_i + \chi_j$. If $i = j$ then this would give proportional Lyapunov functionals χ_i and $2\chi_i$, which is impossible. If $i \neq j$ then this would give a resonance among the Lyapunov functionals χ_i, χ_i, and $\chi_l = \chi_i + \chi_j$, which is also impossible. Hence the Lie group N is abelian and so the infranilmanifold $\mathcal{N}$ is finitely covered by a torus. This completes the proof of Theorem 1.1.

References

[1] Y. Benoist and F. Labourie, *Sur les difféomophismes d'Anosov affines à feuilletages stable et instable différentiables,* Invent. Math. 111 (1993), 285–308.

[2] J. Franks, *Anosov diffeomorphisms on tori,* Trans. Amer. Math. Soc. 145 (1969), 117–124.

[3] E. Goetze and R. Spatzier, *Smooth classification of Cartan actions of higher rank semisimple Lie groups and their lattices,* Ann of Math. (2) 150 (1999), 743–773.

[4] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds,* Lecture Notes in Math., 583, Springer-Verlag, New York, 1977.

[5] S. Hurder, *Rigidity for Anosov actions of higher rank lattices,* Ann. of Math. (2) 135 (1992), 361–410.

[6] J.-L. Journé, *A regularity lemma for functions of several variables,* Rev. Mat. Iberoamericana 4 (1988), 187–193.

[7] B. Kalinin and A. Katok, *Invariant measures for actions of higher rank abelian groups,* Smooth ergodic theory and its applications (Seattle, 1999), Proc. Sympos. Pure Math., 69, pp. 593–637, Amer. Math. Soc., Providence, RI, 2001.

[8] B. Kalinin and V. Sadovskaya, *Global rigidity for totally nonsymplectic Anosov* $\mathbb{Z}^k$ *actions,* Geom. Topol. 10 (2006), 929–954.

[9] B. Kalinin and R. Spatzier, *Rigidity of the measurable structure for algebraic actions of higher rank abelian groups,* Ergodic Theory Dynam. Systems 25 (2005), 175–200.

[10] ———, *On the classification of Cartan actions,* Geom. Funct. Anal. (to appear).

[11] A. Katok and J. Lewis, *Local rigidity for certain groups of toral automorphisms,* Israel J. Math. 75 (1991), 203–241.

[12] A. Katok, V. Nitica, and A. Török, *Non-abelian cohomology of abelian Anosov actions,* Ergodic Theory Dynam. Systems 20 (2000), 259–288.

[13] A. Katok and R. J. Spatzier, *Invariant measures for higher rank hyperbolic Abelian actions*, Ergodic Theory Dynam. Systems 16 (1996), 751–778.

[14] ———, *Differential rigidity of Anosov actions of higher rank abelian groups and algebraic lattice actions,* Tr. Mat. Inst. Steklova 216 (1997), Din. Sist. i Smezhnye Vopr., 292–319; translation in Proc. Steklov Inst. Math. 216 (1997), 287–314.

[15] R. de la Llave, J. M. Marco, and R. Moriyón, *Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation,* Ann. of Math. (2) 123 (1986), 537–611.

[16] A. Manning, *There are no new Anosov diffeomorphisms on tori,* Amer. J. Math. 96 (1974), 422–429.

[17] S. Newhouse, *On codimension one Anosov diffeomorphisms,* Amer. J. Math. 92 (1970), 761–770.

[18] V. Nitica and A. Török, *On the cohomology of Anosov actions,* Rigidity in dynamics and geometry (Cambridge, 2000), pp. 345–361, Springer-Verlag, Berlin, 2002.

[19] J. Palis and J. C. Yoccoz, *Centralizers of Anosov diffeomorphisms on tori,* Ann. Sci. École Norm. Sup. (4) 22 (1989), 99–108.

[20] C. Pugh, M. Shub, and A. Wilkinson, *Partial differentiability of invariant splittings,* J. Statist. Phys. 144 (2004), 891–921.

[21] N. Qian, *Dynamics and ergodic properties of Anosov* $\mathbb{R}^n$ *actions,* Random Comput. Dynam. 2 (1994), 21–40.

[22] F. Rodriguez Hertz, *Global rigidity of certain abelian actions by toral automorphisms,* J. Modern Dynam. (to appear).

[23] V. Sadovskaya, *On uniformly quasiconformal Anosov systems,* Math. Res. Lett. 12 (2005), 425–441.

[24] P. Tukia, *On quasiconformal groups,* J. Anal. Math. 46 (1986), 318–346.

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