

Rigidity of the measurable structure for algebraic actions of higher-rank Abelian groups

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Abstract. We investigate rigidity of measurable structure for higher-rank Abelian algebraic actions. In particular, we show that ergodic measures for these actions fiber over a zero entropy measure with Haar measures along the leaves. We deduce various rigidity theorems for isomorphisms and joinings as corollaries.

1. Introduction

We consider algebraic actions of higher-rank Abelian groups on homogeneous spaces. We establish rigidity properties of the measurable structure for these actions. This is part of a general program investigating such actions and their rigid structures such as local rigidity and triviality of cocycles [13, 14]. As for the measurable structure for Anosov actions without rank-one factors, one conjectures that invariant ergodic measures for such actions are always algebraic, i.e. Haar measures on closed homogeneous subspaces [15, 21]. For the case of measures on the circle invariant under x^2, x^3 this is the famous Furstenberg conjecture [4]. To this date, the only results known in this direction concern the classification of invariant measures with positive entropy. The first such results were achieved by Lyons and later Rudolph and Johnsson for the Furstenberg conjecture [7, 8, 22, 29]. More general results were obtained later by Katok and Spatzier in [15, 16] using geometric ideas.

In this paper we generalize the approach of [15] to general partially hyperbolic algebraic actions assuming that certain one-parameter subgroups have ‘large’ ergodic components. As a main application we show that measurable isomorphisms between two algebraic actions of this type have to be algebraic. This had previously only been proved for the case of a partially hyperbolic \mathbb{Z}^k -action by toral automorphisms [11]. Our result covers actions on nilmanifolds by automorphisms as well as Weyl chamber flows and other actions on homogeneous spaces of higher-rank semisimple groups. We also obtain results on the rigidity of joinings. In particular, joinings in the semisimple case have to be algebraic assuming suitable one-parameter subgroups are ergodic. Previously, rigidity results for

joinings had only been obtained for actions by toral automorphisms [9]. Even in that case, our results apply more generally and resolve some questions raised in [9].

We state the main results for \mathbb{R}^k -actions in §3 and for \mathbb{Z}^k -actions in §4 after establishing basic notions in §2. The proof in the case of an \mathbb{R}^k -action on a homogeneous space G/Γ is given in §§5 and 6. The central idea of the proof is the same as in [15]. Essentially we prove a dichotomy for the conditional measures of an invariant measure μ along various coarse Lyapunov foliations. They are either Haar or atomic. This is done as in [15] by moving these conditional measures by suitable elements in \mathbb{R}^k which act isometrically along these foliations. Now either all these conditional measures are atomic and the entropy is zero or the measure is invariant under unipotent groups. Ratner's theorem then shows that suitable ergodic components of the measure for the unipotent action are Haar measures on compact orbits of some subgroup H of G . We now show that μ essentially lives on an orbit or orbit closure of the normalizer of H . From this we can describe μ as a measure on a fibration with H -orbits and Haar measures as fibers. In the semisimple and nilpotent case, we can additionally show that this quotient is algebraic. The main novelty of our argument lies in this latter part of the proof. The factorization obtained is quite strong and allows us in particular to get general results about isomorphism rigidity and joinings. Our techniques give new results about joinings even for toral automorphisms. We derive the results for measurable isomorphisms and joinings as well as the results for \mathbb{Z}^k -actions in the later sections of the paper.

2. Preliminaries and notations

2.1. *Algebraic \mathbb{R}^k -actions.* Let G be a connected Lie group and $\Gamma \subset G$ be a uniform lattice. Let ρ be an embedding of \mathbb{R}^k into G and let $A = \rho(\mathbb{R}^k) \subset G$ be its image. Then \mathbb{R}^k acts on $\mathcal{M} = G/\Gamma$ via left translations by corresponding elements of A . We denote this action by α . The linear part of α is the adjoint representation of A on the Lie algebra \mathfrak{g} of G . We will assume that for all $a \in A$, $\text{Ad}(a)$ is a diagonalizable (semisimple) automorphism of the Lie algebra \mathfrak{g} . We note that α preserves the Haar measure λ on $\mathcal{M} = G/\Gamma$. We call (α, μ) an algebraic \mathbb{R}^k -action. A prime example of such an action is given by an embedding of \mathbb{R}^k into a split Cartan subgroup of a semisimple Lie group G , for example $\text{SL}(n, \mathbb{R})$ with $n > k$.

Let us briefly describe the relation between the Lyapunov exponents and the algebraic structure of the action. Since A is commutative the Lie algebra \mathfrak{g} splits into invariant subspaces for the adjoint action of A , which are the eigenspaces corresponding to real eigenvalues and invariant subspaces corresponding to the pairs of complex eigenvalues. For each invariant subspace let us denote by $\chi(a)$ the logarithm of the modulus of the corresponding eigenvalue of $\text{Ad}(a)$. Then $\chi(a)$ is a linear functional on A . This splitting gives rise to the right-invariant splitting of the tangent bundle of G , which projects to the α -invariant splitting of the tangent bundle of $\mathcal{M} = G/\Gamma$. The latter is a refinement of the Lyapunov decomposition of $T\mathcal{M}$ into Lyapunov subspaces for the action α . Indeed, it is easy to see that for any vector v in one of the right-invariant distributions its Lyapunov exponent $\chi(\alpha(a), v)$ is equal to the corresponding $\chi(a)$.

In fact it is easy to see that we can take an inner product on \mathfrak{g} such that the invariant subspaces are orthogonal and for any $a \in A$ the restriction of $\text{Ad}(a)$ to each of the invariant

subspaces is a scalar multiple of isometry. This inner product gives rise to the right-invariant Riemannian metric on G , which projects to the Riemannian metric on $\mathcal{M} = G/\Gamma$. It follows that with respect to this metric

$$\|D\alpha(a)v\| = e^{\chi(a)}\|v\|$$

for any $a \in \mathbb{R}^k$ and vector v in the distribution corresponding to χ .

Each Lyapunov exponent for the action α is a linear functional on A . Its positive half-space is called a *Lyapunov half-space* and its boundary a *Lyapunov hyperplane*. Note that Lyapunov exponents may be proportional to each other with positive or negative coefficients. In this case, they define the same Lyapunov hyperplane. Combining Lyapunov distributions with the same Lyapunov half-space we obtain *coarse Lyapunov distributions* which form the *coarse Lyapunov decomposition*.

An element $a \in \mathbb{R}^k$ is called *regular* if it does not belong to any Lyapunov hyperplane. All other elements are called *singular*. Call a singular element *generic* if it belongs to only one Lyapunov hyperplane.

For an element $a \in A$ the *stable*, *unstable* and *neutral* distributions E_a^- , E_a^+ and E_a^0 are defined as the sum of the Lyapunov spaces for which the value of the corresponding Lyapunov exponent on a is negative, positive or zero respectively. From the above discussion we see that the derivative of any singular element a acts isometrically on its neutral distribution. So in this case E_a^0 coincides with the isometric distribution E_a^I .

Notice that while some Lyapunov distributions may not be integrable the stable and unstable distributions always integrate to homogeneous foliations whose leaves are in fact cosets of nilpotent subgroups. For an element a we will denote the integral foliations of the stable and unstable distributions E_a^- and E_a^+ by W_a^- and W_a^+ .

We note that E_a^- is the sum of certain coarse Lyapunov distributions. Conversely, any coarse Lyapunov distribution is the intersection of the strong stable distributions for certain elements of the action. Thus, any coarse Lyapunov distribution integrates to a homogeneous foliation whose leaves are also cosets of a nilpotent subgroup.

2.2. Algebraic \mathbb{Z}^k -actions and their suspensions. Let G be a simply connected nilpotent Lie group and $\Gamma \subset G$ be a uniform lattice. If A is an automorphism of A such that $A(\Gamma) = \Gamma$, then it projects to an automorphism F_A of the nilmanifold $\mathcal{M} = G/\Gamma$ which preserves the Haar measure λ on \mathcal{M} . The differential of A at $e \in G$ is an automorphism of the Lie algebra \mathfrak{g} of G and is called the linear part of A . By a theorem of Parry [24] F_A is ergodic with respect to λ if and only if no eigenvalue of the quotient of the linear part of A on the Abelianization $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a root of unity, furthermore, in this case F_A is a K-automorphism.

If automorphisms A_1, \dots, A_k as above commute, they define a \mathbb{Z}^k -action α by automorphisms F_{A_i} of G/Γ . Such an action we call an algebraic \mathbb{Z}^k -action by automorphisms of the nilmanifold $\mathcal{M} = G/\Gamma$. A prime example of such an action is given by an embedding of \mathbb{Z}^k into $SL(n, \mathbb{Z})$, with $n > k$, whose elements act by automorphisms of \mathbb{R}^n which preserve the lattice \mathbb{Z}^n and thus project to the automorphisms of the torus \mathbb{T}^n . In this paper we will restrict our attention to the actions for which the linear parts of the automorphisms are diagonalizable.

Let us briefly describe the Lyapunov exponents for the algebraic \mathbb{Z}^k action by automorphisms of the nilmanifolds. Since A_1, \dots, A_k commute the Lie algebra \mathfrak{g} splits into invariant subspaces for their linear parts, which are the eigenspaces corresponding to real eigenvalues and invariant subspaces corresponding to the pairs of complex eigenvalues. For each invariant subspace let us denote by $\chi(a)$ the logarithm of the modulus of the corresponding eigenvalue of $\text{Ad}(a)$. Then $\chi(a)$ is an additive functional on \mathbb{Z}^k . This splitting gives rise to the right-invariant splitting of the tangent bundle of G , which projects to the α -invariant splitting of the tangent bundle of $\mathcal{M} = G/\Gamma$. The latter is again a refinement of the Lyapunov decomposition of $T\mathcal{M}$ into Lyapunov subspaces for the action α . Similarly to the case of algebraic \mathbb{R}^k -actions, one can easily construct a Riemannian metric on $\mathcal{M} = G/\Gamma$ with respect to which

$$\|D\alpha(a)v\| = e^{\chi(a)}\|v\|$$

for any $a \in \mathbb{Z}^k$ and vector v in the distribution corresponding to χ .

We would like to view the additive functionals on \mathbb{Z}^k as linear functionals on \mathbb{R}^k and be able to operate with the elements in their kernels, i.e. in the Lyapunov hyperplanes. For this we need to pass from an action of \mathbb{Z}^k to the corresponding action of \mathbb{R}^k via the so-called suspension construction.

Suppose \mathbb{Z}^k acts on G/Γ . Embed \mathbb{Z}^k as a lattice in \mathbb{R}^k . Let \mathbb{Z}^k act on $\mathbb{R}^k \times (G/\Gamma)$ by $z(x, m) = (x - z, z \cdot m)$ and form the quotient

$$N = (\mathbb{R}^k \times (G/\Gamma))/\mathbb{Z}^k.$$

Note that the action of \mathbb{R}^k on $\mathbb{R}^k \times (G/\Gamma)$ by $x \cdot (y, n) = (x + y, n)$ commutes with the \mathbb{Z}^k -action and therefore descends to the solvmanifold N .

The manifold N is a fibration over the ‘time’ torus \mathbb{T}^k with the fiber G/Γ . We note that TN splits into the direct sum $TN = T_f N \oplus T_o N$ where $T_f N$ is the subbundle tangent to the G/Γ fibers and $T_o N$ is the subbundle tangent to the orbit foliation. The Lyapunov exponent corresponding to $T_o N$ is always identically zero. To exclude this trivial case, we will consider only the Lyapunov exponents corresponding to $T_f N$. These Lyapunov exponents of the \mathbb{R}^k -action are the extensions of the Lyapunov exponents of the \mathbb{Z}^k -action to the linear functionals on \mathbb{R}^k .

We note that the suspension of an algebraic \mathbb{Z}^k -action is an algebraic \mathbb{R}^k -action. The further definitions and basic properties related to the Lyapunov exponents, distributions and foliations are the same as in the case of algebraic \mathbb{R}^k -actions. We call an action *totally non-symplectic* (TNS) if no Lyapunov exponents are proportional with negative coefficients.

Note that any \mathbb{Z}^k -invariant measure on G/Γ lifts to a unique \mathbb{R}^k -invariant measure on N and conversely any invariant measure for the suspension induces a unique invariant measure for the original action.

2.3. Measure-theoretic notations. Let α be a \mathbb{Z}^k - or \mathbb{R}^k -action on a compact metric space. For any α -invariant Borel probability measure μ we denote by ξ_a the partition into ergodic components of the element $\alpha(a)$ and by $h_\mu(a)$ the measure-theoretic entropy of the map $\alpha(a)$. If W is a continuous foliation of the metric space, $\xi(W)$ denotes its

measurable hull, i.e. the measurable partition whose elements consist μ -almost everywhere of complete leaves of W .

Suppose α and α' are measurable actions of the same group T by measure-preserving transformations of the spaces (X, λ) and (Y, ν) , respectively. A probability measure μ on $X \times Y$ is called a *joining* of α and α' if it is invariant under the diagonal (product) action $\alpha \times \alpha'$ of T on $X \times Y$ and its projections to X and Y coincide with λ and ν , respectively. The product measure $\mu = \lambda \times \nu$ is called the trivial joining, and all other joining measures are called non-trivial joinings.

A particular case of a joining is given by a measurable isomorphism, i.e. a measurable map ϕ which intertwines the actions α and α' . In fact, the push-forward of λ under the graph of ϕ , $(\text{id} \times \phi)_*(\lambda)$, is a joining. We call a measurable isomorphism ϕ between two algebraic \mathbb{Z}^k -actions α and α' *algebraic* if ϕ lifts to an affine isomorphism ψ between the ambient groups, i.e. ψ is a composite of an isomorphism with left translations.

For a probability measure μ on G/Γ we denote by $\Lambda(\mu)$ the subgroup of all elements in G whose left translations preserve μ and by $L(\mu)$ the Lie algebra of $\Lambda(\mu)$. The measure μ is called *algebraic* if there exists $x \in G/\Gamma$ such that $\mu(x\Lambda(\mu)) = 1$, i.e. μ is the Haar measure on a single left coset of $\Lambda(\mu)$.

3. Main results

In this section we formulate our main results on rigidity of invariant measures, measure-preserving isomorphisms, and joinings for homogeneous \mathbb{R}^k -actions.

3.1. Rigidity of invariant measures.

THEOREM 3.1. *Let G be a connected Lie group, $\Gamma \subset G$ be a uniform lattice, and A be a subgroup of G isomorphic to \mathbb{R}^k , with $k \geq 2$, such that for all $a \in A$, $\text{Ad}(a)$ is a diagonalizable-over- \mathbb{C} automorphism of the Lie algebra \mathfrak{g} . Let α denote the A -action on G/Γ by left translations, and let μ be an ergodic invariant measure for α . Suppose further that either the eigenvalues of $\text{Ad}(a)$ are real for all $a \in A$ or μ is weakly mixing for α .*

Suppose for some regular element $b \in A$ we have that for each coarse Lyapunov subfoliation $W^P \subset W_b^-$ there exists a singular element a in the corresponding Lyapunov hyperplane ∂P which acts ergodically on W^P , i.e. $\xi_a \leq \xi(W^P)$.

Then there exists a subgroup $H \subset G$ such that:

- (1) *A normalizes H , i.e. $aHa^{-1} = H$ for any $a \in A$;*
- (2) *$H \subset \Lambda(\mu)$, i.e. left translations by elements of H leave μ invariant;*
- (3) *for almost every leaf of W_b^- the conditional measure of μ on this leaf is Haar on a single left coset of the Lie subgroup of G with Lie algebra $E_b^- \cap \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of H ;*
- (4) *for μ -almost every x the coset Hx is compact in M , and the ergodic components of μ with respect to the left action of H are Haar measures on the left cosets of H . Moreover, α induces a measurable factor action of A on the space of these ergodic components. The entropy of the factor measure is zero with respect to any element in the Weyl chamber of b ;*

- (5) *if, in addition, the coset $N_G(H)y$ of the normalizer of H in G is compact for some y whose A orbit is dense in $\text{supp}(\mu)$, then $\text{supp}(\mu) \subset N_G(H)y$ and for any $x \in N_G(H)y$ the coset Hx is compact in G/Γ . In this case, the restriction of α to $N_G(H)y$ induces an algebraic factor action which is algebraically isomorphic to the action of $y^{-1}Ay$ on $(y^{-1}Hy) \setminus y^{-1}N_G(H)y/\Gamma$. The entropy of the factor measure is zero with respect to any element in A and the conditional measures in the fibers are Haar. Also, conclusion (3). holds for the stable foliation W_c^- of any element $c \in A$.*

We prove this theorem in §§5 and 6.

Remark 1. If H is a normal subgroup then $N_G(H)y = Gy$ and the last conclusion of the theorem above applies.

Remark 2. It is easy to see that the measure μ in the theorem is algebraic, i.e. is supported on a single coset of H , if and only if $A \subset H$, in other words if H is α -invariant.

Theorem 3.1 allows us to obtain some important corollaries, which include the isomorphism rigidity theorem in §3.4.

COROLLARY 3.2. *Suppose, in addition to the assumptions of Theorem 3.1, that for some element c in the Weyl chamber of b , $\alpha(c)$ is a K -automorphism with respect to the measure μ . Then μ is an algebraic measure, i.e. it is Haar on a single left coset Hx .*

Proof. Since $\alpha(c)$ is a K -automorphism with respect to the measure μ , the measurable factor with zero entropy has to be trivial. Hence μ is Haar on a single left coset Hx . \square

COROLLARY 3.3. (Semisimple case) *Let G be a semisimple group and $\Gamma \subset G$ be a uniform lattice. Let α be an \mathbb{R}^k -action, $k \geq 2$, on G/Γ by left translations given by an embedding ρ of \mathbb{R}^k into a split Cartan subgroup of G . Let μ be an ergodic invariant measure for α .*

Suppose for some regular element $b \in \mathbb{R}^k$ we have that for each coarse Lyapunov subfoliation $W^P \subset W_b^-$ there exists a singular element a in the corresponding Lyapunov hyperplane ∂P which acts ergodically on W^P , i.e. $\xi_a \leq \xi(W^P)$. Then the conclusions (1)–(5) of Theorem 3.1 hold.

We prove this corollary in §7.

While the ergodicity assumption $\xi_a \leq \xi(W^P)$ in Theorem 3.1 and Corollary 3.3, and K -automorphism assumption in Corollary 3.2 are somewhat restrictive, they can be verified for a certain type of joining measure associated with measure-preserving isomorphisms. This allows us to deduce the following results on the rigidity of measure-preserving isomorphisms.

3.2. Isomorphism rigidity and rigidity of joinings. We note that the ergodicity and K -automorphism assumptions in the following theorem are with respect to the Haar measure. Therefore they can be checked in various cases as Corollary 3.5 illustrates.

THEOREM 3.4. *Let G be a connected Lie group and $\Gamma \subset G$ be a uniform lattice. Let ρ be an embedding of \mathbb{R}^k into G such that for all $a \in \mathbb{R}^k$, $\text{Ad}(\rho(a))$ is a diagonalizable-over- \mathbb{C} automorphism of the Lie algebra \mathfrak{g} . Let α denote the corresponding \mathbb{R}^k -action on G/Γ by left translations, and let λ be the Haar measure on G/Γ . Let (α', λ') be an \mathbb{R}^k -action of the same type on G'/Γ' .*

Suppose that \mathbb{R}^k contains a subgroup S isomorphic to \mathbb{R}^2 for which:

- (1) *the α -action of any one-parameter subgroup is ergodic with respect to λ ;*
- (2) *there exists an element $b \in S$ such that $\alpha(b)$ is a K -automorphism with respect to λ .*

Then any measure-preserving isomorphism $h : G/\Gamma \rightarrow G'/\Gamma'$ between (α, λ) and (α', λ') coincides (mod 0) with an affine isomorphism, and hence α and α' are algebraically isomorphic.

Proof. Consider the diagonal action $\alpha \times \alpha'$ of \mathbb{R}^k on $G/\Gamma \times G'/\Gamma'$. Let μ be the push-forward of λ to the graph of the measure-preserving isomorphism h . Then μ is an invariant measure for $\alpha \times \alpha'$, and $(\alpha \times \alpha', \mu)$ has the same ergodic properties as (α, λ) . Hence the assumptions of Theorem 3.1 and Corollary 3.2 are satisfied for the restriction of $\alpha \times \alpha'$ to the subgroup $S \subset \mathbb{R}^k$. Thus μ is Haar on a single left coset Hx . Since h is a measure-preserving isomorphism and μ projects to the Haar measures in the factors, we see that the coset Hx has to project one-to-one onto the factors. Thus Hx is a graph of an affine isomorphism \tilde{h} between G/Γ and G'/Γ' which coincides (mod 0) with h . \square

COROLLARY 3.5. (Semisimple case) *Let G be a semisimple group without compact factors, $\Gamma \subset G$ be an irreducible uniform lattice, and λ be the Haar measure on G/Γ . Let (α, λ) be an \mathbb{R}^k -action, $k \geq 2$, on G/Γ by left translations given by an embedding ρ of \mathbb{R}^k into a split Cartan subgroup of G .*

Let G' be any simply connected Lie group and $\Gamma' \subset G'$ be a uniform lattice. Let ρ' be an embedding of \mathbb{R}^k into G' such that for all $a \in \mathbb{R}^k$, $\text{Ad}(\rho'(a))$ is a diagonalizable-over- \mathbb{C} automorphism of the Lie algebra \mathfrak{g}' . Let α' denote the corresponding \mathbb{R}^k -action on G'/Γ' by left translations, and let λ' be the Haar measure on G'/Γ' . Then any measure-preserving isomorphism $h : G/\Gamma \rightarrow G'/\Gamma'$ between (α, λ) and (α', λ') coincides (mod 0) with an affine isomorphism, and hence α and α' are algebraically isomorphic.

Proof. We apply Theorem 3.4. It suffices to check that the restriction of α to any one-parameter subgroup of \mathbb{R}^k is ergodic with respect to λ and that there exists an element $b \in \mathbb{R}^k$ such that $\alpha(b)$ is a K -automorphism with respect to λ . The first follows from the Howe–Moore theorem [14] and the latter is proved in [2]. \square

In the proof of Theorem 3.4 we applied the main Theorem 3.1 to a special type of joining measure. The following theorem describes the rigidity of joinings in general. Here, however, we make the ergodicity assumption for the joining measure itself.

THEOREM 3.6. (Semisimple case) *Let G_1 be a semisimple group, $\Gamma_1 \subset G_1$ be a cocompact lattice, and λ_1 be the Haar measure on G_1/Γ_1 . Let (α_1, λ_1) be an \mathbb{R}^k -action, $k \geq 2$, on G_1/Γ_1 by left translations given by an embedding ρ_1 of \mathbb{R}^k into a split Cartan subgroup of G_1 . Suppose that $\alpha_1(a)$ is a K -automorphism with respect to λ_1 for some $a \in \mathbb{R}^k$.*

Let (α_2, λ_2) be an \mathbb{R}^k -action of the same type on G_2/Γ_2 , and let $(G_1/\Gamma_1 \times G_2/\Gamma_2, \alpha_1 \times \alpha_2, \mu)$ be a non-trivial joining of the actions α_1 and α_2 . If all one-parameter subgroups of \mathbb{R}^k act ergodically with respect to the joining measure μ , then μ is algebraic and the actions α_1 and α_2 have algebraic factors whose finite covers are algebraically isomorphic.

We prove this theorem in §9.

4. \mathbb{Z}^k -actions by automorphisms of nilmanifolds

In this section we state the counterparts of the rigidity results of §3 in the context of \mathbb{Z}^k -actions by automorphisms of tori and nilmanifolds.

4.1. *Rigidity of invariant measures.* We will consider actions of \mathbb{Z}^k on nilmanifolds G/Γ by automorphisms. If the automorphisms are diagonalizable, we can extend the homomorphism of \mathbb{Z}^k into the automorphisms of G to \mathbb{R}^k . This allows us to consider the induced action in the next theorem.

THEOREM 4.1. *Let α be a \mathbb{Z}^k -action, $k \geq 2$, by automorphisms of G/Γ , where G is a simply connected nilpotent Lie group and $\Gamma \subset G$ is a cocompact lattice. Let μ be an ergodic invariant measure for α . Suppose that the corresponding automorphisms of the Lie algebra \mathfrak{g} are diagonalizable over \mathbb{C} , and either all their eigenvalues are real or μ is a weakly mixing measure for α .*

Suppose for some regular element b for the suspension action of \mathbb{R}^k we have that for each coarse Lyapunov subfoliation $W^P \subset W_b^-$ there exists a singular element a in the corresponding Lyapunov hyperplane ∂P which acts ergodically on W^P , i.e. $\xi_a \leq \xi(W^P)$.

Then there exists a subgroup $H \subset G$ such that:

- (1) H is α -invariant;
- (2) $H \subset \Lambda(\mu)$, i.e. left translations by elements of H leave μ invariant;
- (3) for any element $c \in \mathbb{Z}^k$ and for almost every leaf of W_c^- the conditional measure of μ on this leaf is Haar on a single left coset of the Lie subgroup of G with Lie algebra $E_c^- \cap \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of H ;
- (4) μ is supported on a coset $N_G(H)y$ of the normalizer of H in G . For any $x \in N_G(H)y$ the coset Hx is compact in G/Γ , and the restriction of α to $N_G(H)y$ induces an algebraic factor action which is isomorphic to $(y^{-1}Hy) \setminus y^{-1}N_G(H)y/\Gamma$. The entropy of the factor measure is zero with respect to any element in \mathbb{Z}^k and the conditional measures in the fibers are Haar.

This theorem is proved in §8. It generalizes the earlier results for the actions by toral automorphisms, see Theorem 5.1' in [16] and its modified form Theorem 3.1 in [10]. In the toral case, however, these theorems make no assumption on the corresponding automorphisms of the Lie algebra. While technically complicated, one can probably allow the presence of Jordan blocks and thus remove the diagonalizability assumption in the nilmanifold case. The presence of complex eigenvalues in the absence of weak mixing is overcome in the toral case by, possibly, splitting the measure into finitely many components. However, it is not clear whether the corresponding arguments in the proof of Theorem 3.1 in [10] can be generalized to the non-Abelian case.

In the case of actions by automorphisms of nilmanifolds the ergodicity assumption $\xi_a \leq \xi(W^P)$ in Theorem 4.1 can be verified for an arbitrary measure in the important special case of TNS actions, i.e. the partially hyperbolic actions with no negatively proportional Lyapunov exponents. This is illustrated by the following corollary (cf. [15]).

COROLLARY 4.2. (TNS case) *Let α be a TNS \mathbb{Z}^k -action by automorphisms of G/Γ , where G is a simply connected nilpotent Lie group and Γ is a cocompact lattice. Let μ be an ergodic invariant measure for α . Suppose that the corresponding automorphisms of the Lie algebra \mathfrak{g} are diagonalizable, and either all their eigenvalues are real or μ is a weakly mixing measure for α .*

Then the conclusions (1)–(4) of Theorem 4.1 hold.

Proof. To apply Theorem 4.1 we need to check for some regular element $b \in \mathbb{R}^k$ in the suspension of α that for each coarse Lyapunov subfoliation $W^P \subset W_b^-$ there exists a singular element a in the corresponding Lyapunov hyperplane ∂P which acts ergodically on W^P , i.e. $\xi_a \leq \xi(W^P)$. In the case of TNS actions this is actually true for any regular element b and any generic singular element $a \in \partial P$.

Consider a generic singular element $a \in \partial P$. Take a regular element c in the complement of P so close to a that it is not separated from a by any Lyapunov hyperplane. Then all Lyapunov exponents that are non-zero for a have the same signs on a and c . Since the action is TNS, all Lyapunov exponents that correspond to W^P become positive on c . Thus $W_c^+ = W_a^+ \oplus W^P$ and $W_c^- = W_a^-$. Birkhoff averages with respect to $\alpha(a)$ of any continuous function are constant on the leaves of ξ_a . Since such averages generate the algebra of $\alpha(a)$ -invariant functions we conclude that $\xi_a \leq \xi(W_a^-)$. On the other hand both $\xi(W_c^-)$ and $\xi(W_c^+)$ coincide with the Pinsker algebra $\pi(\alpha(c))$ [19, Theorem B]. Thus we conclude

$$\xi_a \leq \xi(W_a^-) = \xi(W_c^-) = \pi(\alpha(c)) = \xi(W_c^+) = \xi(W_a^+ \oplus W^P) \leq \xi(E^P). \quad \square$$

COROLLARY 4.3. *Suppose, in addition to the assumptions of Theorem 4.1 or Corollary 4.2, that $\alpha(c)$ is a K-automorphism with respect to the measure μ for some element $c \in \mathbb{Z}^k$. Then μ is an algebraic measure, i.e. it is Haar on a single left coset Hx .*

We note that under K-automorphism assumption μ is automatically weakly mixing, so that complex eigenvalues are allowed with no extra assumption.

Proof. Since $\alpha(c)$ is a K-automorphism with respect to the measure μ , the factor with zero entropy has to be trivial. Hence μ is Haar on a single left coset Hx . \square

4.2. Isomorphism rigidity and rigidity of joinings. As in the case of the homogeneous \mathbb{R}^k -actions, we apply the previous theorems to obtain the following results on the isomorphism rigidity and rigidity of joinings.

THEOREM 4.4. *Let α be a \mathbb{Z}^k -action by automorphisms of G/Γ , where G is a simply connected nilpotent Lie group and $\Gamma \subset G$ is a cocompact lattice. Let λ be the Haar measure on G/Γ . Let (α', λ') be a \mathbb{Z}^k -action of the same type on G'/Γ' .*

Suppose there is a subgroup $S \subset \mathbb{Z}^k$ isomorphic to \mathbb{Z}^2 such that $\alpha(a)$ is ergodic with respect to λ for all $a \in S$. Then any measure-preserving isomorphism $h : G/\Gamma \rightarrow G'/\Gamma'$

between (α, λ) and (α', λ') coincides (mod 0) with an affine isomorphism, and hence α and α' are algebraically isomorphic.

Proof. The proof of this theorem uses Corollary 4.3 and closely follows the proofs of Theorem 3.4 and Corollary 3.5.

Consider the suspension actions $\tilde{\alpha}$ and $\tilde{\alpha}'$ of \mathbb{R}^k on the corresponding solvmanifolds \mathcal{M} and \mathcal{M}' . By assumption, the restriction of α to $S \subset \mathbb{Z}^k$ acts by ergodic automorphisms of G/Γ . Then the proof of Corollary 4 in [14] shows that the ergodicity assumption $\xi_a \leq \xi(W^P)$ of Theorem 4.1 is satisfied for the restriction of $\tilde{\alpha}$ to the linear hull $\tilde{S} \subset \mathbb{R}^k$ of S . We recall that any ergodic automorphism of G/Γ is a K-automorphism [24].

Consider the diagonal (product) action $\tilde{\alpha} \times \tilde{\alpha}'$ of \mathbb{R}^k on $\mathcal{M} \times \mathcal{M}'$. The measure-preserving isomorphism $h : G/\Gamma \rightarrow G'/\Gamma'$ between (α, λ) and (α', λ') can be lifted to a measure-preserving isomorphism \tilde{h} between $\tilde{\alpha}$ and $\tilde{\alpha}'$. Let $\tilde{\mu}$ be the push-forward of the invariant measure on \mathcal{M} to the graph of the measure-preserving isomorphism \tilde{h} . Then $\tilde{\mu}$ is an invariant measure for $\tilde{\alpha} \times \tilde{\alpha}'$ and can be viewed as the suspension of the invariant measure μ for $\alpha \times \alpha'$ which is the push-forward of λ to the graph of h in $G/\Gamma \times G'/\Gamma'$.

Since $(\alpha \times \alpha', \tilde{\mu})$ has the same ergodic properties as $\tilde{\alpha}$, the assumptions of Theorem 4.1 and Corollary 4.3 are satisfied for the measure μ invariant under the restriction of $\alpha \times \alpha'$ to the subgroup $S \subset \mathbb{Z}^k$. Thus μ is Haar on a single left coset $Hx \subset G/\Gamma \times G'/\Gamma'$. Since h is a measure-preserving isomorphism and μ projects to the Haar measures on the factors, we see that the coset Hx has to project one-to-one onto the factors. Thus Hx is a graph of an affine isomorphism h' between G/Γ and G'/Γ' which coincides (mod 0) with h . \square

THEOREM 4.5. *Let α_1 be a \mathbb{Z}^k -action, $k \geq 2$, by automorphisms of G_1/Γ_1 , where G_1 is a simply connected nilpotent Lie group and $\Gamma_1 \subset G_1$ is a cocompact lattice. Suppose that the corresponding automorphisms of the Lie algebra \mathfrak{g}_1 are diagonalizable over \mathbb{C} and suppose that there exists such an element $a \in \mathbb{Z}^k$ that $\alpha_1(a)$ is an ergodic automorphism with respect to the Haar measure λ_1 on G_1/Γ_1 . Let (α_2, λ_2) be a \mathbb{Z}^k -action on G_2/Γ_2 of the same type.*

Let $(G_1/\Gamma_1 \times G_2/\Gamma_2, \alpha_1 \times \alpha_2, \mu)$ be a non-trivial joining of the actions α_1 and α_2 . Suppose that either the eigenvalues of the corresponding automorphisms of $G_1/\Gamma_1 \times G_2/\Gamma_2$ are all real, or μ is weakly mixing.

If for any one-parameter subgroup of \mathbb{R}^k , the ergodic components of the suspension of the joining measure μ contain the fibers of the suspension, then the actions α_1 and α_2 have algebraic factors which have algebraically isomorphic finite covers. Moreover, the joining measure μ is an extension of a zero-entropy measure for a common measurable factor with Haar measures in fibers.

We prove this theorem in §9.

COROLLARY 4.6. (TNS case) *Let α_1 be a TNS \mathbb{Z}^k -action by automorphisms of G_1/Γ_1 , where G_1 is a simply connected nilpotent Lie group and $\Gamma_1 \subset G_1$ is a cocompact lattice. Suppose that the corresponding automorphisms of the Lie algebra \mathfrak{g}_1 are diagonalizable over \mathbb{C} and suppose that there exists an element $a \in \mathbb{Z}^k$ such that $\alpha_1(a)$ is an ergodic*

automorphism with respect to the Haar measure λ_1 on G_1/Γ_1 . Let (α_2, λ_2) be a \mathbb{Z}^k -action on G_2/Γ_2 of the same type.

Let $(G_1/\Gamma_1 \times G_2/\Gamma_2, \alpha_1 \times \alpha_2, \mu)$ be a non-trivial joining of the actions α_1 and α_2 . Suppose that either the eigenvalues of the corresponding automorphisms of $G_1/\Gamma_1 \times G_2/\Gamma_2$ are all real, or μ is weakly mixing. Then the actions α_1 and α_2 have algebraic factors which are algebraically isomorphic. Moreover, the joining measure μ is an extension of a zero-entropy measure for a common measurable factor with Haar measures in fibers.

Proof. This corollary follows from Theorem 4.5 similarly to the way Corollary 4.2 follows from Theorem 4.1 (cf. [9]). \square

In the case of actions by automorphisms of tori the previous two results can be slightly strengthened using the stronger theorem on invariant measures for the toral case, see [16, Theorem 5.1'] and [10, Theorem 3.1]. The proof of Theorem 4.5 yields in the toral case Theorem 4.7 below. This theorem generalizes Theorem 3.1 in [9] and gives partial answer to the open problems posed in [9]. Instead of the TNS assumption in Theorem 4.7 we may assume $k \geq 2$ and the ergodicity of one-parameter subgroups for the joining measure.

THEOREM 4.7. *Let (α_1, λ_1) and (α_2, λ_2) be TNS actions of \mathbb{Z}^k by automorphisms of \mathbb{T}^{m_1} and \mathbb{T}^{m_2} , where λ_1 and λ_2 are Lebesgue measures on \mathbb{T}^{m_1} and \mathbb{T}^{m_2} correspondingly. Suppose that there exists an element $a \in \mathbb{Z}^k$ such that $\alpha_1(a)$ is ergodic with respect to λ_1 . If there exists a non-trivial joining measure μ on $\mathbb{T}^m = \mathbb{T}^{m_1} \times \mathbb{T}^{m_2}$ then there exists a subgroup $\Gamma \subset \mathbb{Z}^k$ of finite index such that the actions α_1 and α_2 restricted to Γ have algebraic factors which are conjugate over \mathbb{Q} .*

Moreover, if the joining measure μ is ergodic it decomposes as $(1/N)(\mu_1 + \dots + \mu_N)$, where each μ_i , $i = 1, \dots, N$, is an invariant measure for the restriction of $\alpha_1 \times \alpha_2$ to Γ and is an extension of a zero-entropy measure for the corresponding algebraic factor with Haar measures in the fibers.

5. Proof of Theorem 3.1

5.1. *Scheme of the proof of Theorem 3.1.* We will say that an integrable invariant distribution E and its foliation F satisfy property (A) if there exists a Lie subgroup $L \subset G$ such that for μ -almost every x the conditional measure μ_x^F is a Haar measure on a single left coset Lx .

The main part of the proof is to show that W_b^- satisfies property (A). This is done in §6. After this we prove in §5.2 the existence of the subgroup H which satisfies the first three conclusions of the theorem. Then in §§5.3 and 5.4 we describe the properties of the action in the factor and thus complete the proof of the theorem.

5.2. *The construction and properties of H .* Let Λ_0 be the connected component of the identity of $\Lambda(\mu)$, and let $M \subset \Lambda_0$ be the maximal Lie subgroup of Λ_0 generated by unipotent subgroups of Λ_0 . Since W_b^- satisfies property (A) measure μ is invariant under left translations by the corresponding Lie group L and thus L is contained in M . Let μ_x denote the ergodic components of μ with respect to the left action of M . By Ratner's

theorem [27, Corollary C] applied to M , each μ_x is algebraic, i.e. μ_x is a Haar measure on some closed orbit $H_x \cdot x$ of some subgroup H_x of G which contains M .

Since $A \subset \Lambda_0$ and the adjoint action of A maps unipotent elements to unipotent elements, we see that M is normalized by A . Hence the left action of A preserves the above ergodic decomposition, i.e.,

$$a\mu_x = \mu_{ax}.$$

In particular, a maps the support of μ_x to the support of μ_{ax} . Hence $aH_x x = H_{ax} ax$ and thus

$$H_{ax} = aH_x a^{-1}. \quad (1)$$

We will now show that $H_x = H$ is μ -almost every constant. First we note that since A acts ergodically (1) implies that $\dim(H_x) = d$ is μ -almost every constant. So we can consider the measurable function ϕ from G/Γ to the Grassmanian G_d of d -planes in \mathfrak{g} given by $x \mapsto \mathfrak{h}_x$, where \mathfrak{h}_x is the Lie algebra of H_x . By (1) we have $\phi(ax) = a\phi(x)a^{-1} = \text{Ad}(a)\phi(x)$. Then $\nu = \phi_*(\mu)$ is an ergodic measure on G_d with respect to the adjoint action of A . To show that $H_x = H$ is μ -almost every constant we will prove that ν is a δ -measure.

Since for all $a \in A$, $\text{Ad}(a)$ is a diagonalizable-over- \mathbb{C} automorphism of the Lie algebra \mathfrak{g} , \mathfrak{g} splits into the direct sum of α -invariant subspaces, which are the eigenspaces corresponding to real eigenvalues and invariant two-dimensional subspaces corresponding to the pairs of complex eigenvalues. Consider a regular element c for which different Lyapunov exponents take different values. The support of ν is contained in the set of non-wandering points of $\text{Ad}(c)$ on G_d . It is easy to see that this set consists of d -planes that are spanned by their intersections with the invariant subspaces of the above splitting. If eigenvalues of $\text{Ad}(a)$ are real for all $a \in A$ then clearly all such d -planes are fixed by the adjoint action of A . Hence the ergodic measure ν which is supported on this set must be concentrated on a single point. If complex eigenvalues are present, the set of non-wandering points of $\text{Ad}(c)$ may include invariant tori on which the adjoint action of A is represented by translations. However, since in this case μ is assumed to be weakly mixing for α we see that ν can not be supported on such tori.

Thus we conclude that $H_x = H$ is μ -almost every constant. Now (1) implies that $H_x = H_{ax} = aH_x a^{-1}$ for μ -almost every x and $a \in A$, i.e. A normalizes H . This proves the first conclusion of Theorem 3.1.

Since the ergodic components of μ with respect to M are Haar measures on left cosets of H we see that μ is invariant under left translations by H and thus $H \subset \Lambda(\mu)$. This shows the second conclusion of Theorem 3.1.

Since L is contained in $M \subset H$, its Lie algebra is contained in $E_b^- \cap \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of H . But it clearly can not be strictly contained in $E_b^- \cap \mathfrak{h}$, for otherwise the conditional measures on leaves of W_b^- could not be supported on a single left coset of L . This proves the third conclusion of Theorem 3.1.

5.3. Measurable factor with zero entropy. We have established above that for μ -almost every x the ergodic component μ_x of μ with respect to M is a Haar measure on the compact left coset Hx . We note that this μ_x is also the ergodic component of μ with respect to H .

Since H is normalized by A the left action of A preserves this ergodic decomposition. Hence α induces a measurable factor action of A on the space of the ergodic components of μ for the left action of H . Thus to establish the fourth conclusion of Theorem 3.1 it remains for us to prove that the entropy of the factor measure is zero with respect to any element in the Weyl chamber of b .

To prove that the entropy is zero we show that the measurable factor can be embedded into a topological factor of the restriction of α to the support of μ and then use Proposition 4.1 in [15].

We first show that the volume of cosets Hx is constant almost everywhere. The fact that the left action of A preserves the above ergodic decomposition means that

$$a_*\mu_x = \mu_{ax} \quad (2)$$

where μ_x is the Haar measure on Hx normalized to be probability. We fix some (bi-invariant) Haar measure on H and denote the corresponding measure on Hx by ν_x . Since both ν_x and μ_x are Haar measures $\nu_x = \rho(x)\mu_x$, where $\rho(x)$ is an almost-everywhere-finite measurable function. Since $a_*\nu_x = |\det(\text{Ad}(a))|\nu_{ax}$ we obtain using (2) that $\rho(ax) = |\det(\text{Ad}(a))|\rho(x)$. Using recurrence under a we conclude that $|\det(\text{Ad}(a))| = 1$ μ -almost everywhere. This implies that $\rho(x)$ is A -invariant, hence $\text{Vol}(Hx) = \nu_x(Hx) = \rho(x)$ is constant almost everywhere. We denote this constant by V .

Now we show that for any $x \in \text{supp}(\mu) \subset \mathcal{M} = G/\Gamma$ the coset Hx has finite volume. For any $x \in \mathcal{M}$ the coset Hx , as an orbit of the left action of H , is isomorphic to

$$H/\text{stab}(x) \cong H/H \cap x\Gamma x^{-1}.$$

Here we slightly abuse notation by writing x in $x\Gamma x^{-1}$ instead of an element in G which projects to x . This is justified since $x\Gamma x^{-1}$ does not depend on the choice of such an element.

For any x in the support of μ there exists a sequence of typical points x_n converging to x for which $\text{Vol}(H/H \cap x_n\Gamma x_n^{-1}) = \text{Vol}(Hx_n) = V$. It is easy to see that the sequence of lattices $H \cap x_n\Gamma x_n^{-1}$ converges to a discrete subgroup Γ' of H which lies in $H \cap x\Gamma x^{-1}$. By Chabauty's theorem [25],

$$\text{Vol}(Hx) = \text{Vol}(H/H \cap x\Gamma x^{-1}) \leq \text{Vol}(H/\Gamma') \leq \liminf \text{Vol}(H/H \cap x_n\Gamma x_n^{-1}) = V.$$

Thus for any x in the support of μ the coset Hx has volume bounded above by V . Now we show that it is compact. Since G/Γ is compact, there exists a small open neighborhood U of $e \in G$ such that $U \cdot U^{-1} \cap y\Gamma y^{-1} = \{e\}$ for any $y \in G$. Hence $U \cdot U^{-1} \cap \text{stab}(y) = U \cdot U^{-1} \cap H \cap y\Gamma y^{-1} = \{e\}$. This implies that for any $y \in \mathcal{M}$ the projection from H to $H/H \cap y\Gamma y^{-1}$ is injective on $H \cap U$. Hence for any coset Hx and point $y \in Hx$ there is a fixed-size neighborhood of y in Hx which carries a fixed volume $\text{Vol}(H \cap U)$.

Since for all x in the support of μ the coset Hx has volume bounded above by V we conclude that Hx has to be compact. Moreover, the diameter of these cosets (with respect to the intrinsic metric) is bounded uniformly in $x \in \text{supp}(\mu)$.

Now we can consider the projection π given by $\pi(x) = Hx$, from the support of μ to the space X of all compact subsets of $\mathcal{M} = G/\Gamma$ equipped with the Hausdorff metric.

Since the (intrinsic) diameters of these cosets are uniformly bounded in x it is easy to see that π is Lipschitz continuous.

Let us denote by F and G the projections to X of the foliations W_b^- and $W_b^+ \oplus W_b^0$ respectively. Then F is strictly contracted by any element c in the Weyl chamber of b . Since for μ -almost every $x \in \mathcal{M}$ the conditional measure of μ on the leaf $W_b^-(x)$ is Haar on a single left coset Lx , where $L \subset H$, we conclude that the conditional measures of the factor measure $\pi_*(\mu)$ on the leaves of F are atomic almost everywhere. Now Proposition 4.1 in [15] implies that the entropy of the action in the factor is zero for any element c in the Weyl chamber of b . This completes the proof of the fourth conclusion of Theorem 3.1.

5.4. *Algebraic factor with zero entropy.* The last conclusion of Theorem 3.1 describes the case when the measurable factor obtained above carries a natural algebraic structure. This is always the case when the coset $N_G(H)y$ of the normalizer of H in G is compact for some ‘typical’ point y whose α -orbit is dense in $\text{supp}(\mu)$. Indeed, since A normalizes H the orbit $\alpha(y)$ is contained in $N_G(H)y$. If the closure of the orbit $\alpha(y)$ is $\text{supp}(\mu)$ and if $N_G(H)y$ is compact then clearly $\text{supp}(\mu) \subset N_G(H)y$. In this case, the restriction of α to $N_G(H)y$ has an algebraic factor action isomorphic to $(y^{-1}Hy) \setminus y^{-1}N_G(H)y/\Gamma$.

To complete the proof of the last conclusion of Theorem 3.1 it remains for us to note that the entropy of the factor measure is zero with respect to *any* element in A . The fourth conclusion of Theorem 3.1 tells us that the entropy is zero for any element in the Weyl chamber of b . Clearly, the same is true for any element in the Weyl chamber of b^{-1} . Since any element in A can be represented as a convex combination of elements in these Weyl chambers, the statement follows from subadditivity of entropy for smooth actions established in [6].

Since for any element $c \in A$ the entropy in the factor is zero, the conditional measures on its stable foliation in this factor are atomic ([15], Proposition 4.1). This implies that its stable foliation for the original action satisfies property (A). Thus we have proved that conclusion (3) of the theorem now holds for the stable foliation W_c^- of *any* element $c \in A$.

To complete the proof of Theorem 3.1 it remains for us to show that W_b^- satisfies property (A), which is done in the next section.

6. W_b^- satisfies property (A)

The proof of the fact that W_b^- satisfies property (A) has two main steps. First we show that each coarse Lyapunov subfoliation of W_b^- satisfies property (A). From this we deduce in the second step that W_b^- also satisfies property (A).

6.1. *Step 1.* In this section we show that each coarse Lyapunov subfoliation of W_b^- satisfies property (A).

Let us consider one of the coarse Lyapunov subfoliations $W^P \subset W_b^-$, where P is the corresponding Lyapunov half-space that contains b . By the assumption of Theorem 3.1 there exists a singular element a in the Lyapunov hyperplane ∂P which acts ergodically on W^P , i.e. $\xi_{a_i} \leq \xi(W^P)$. Recall that we denote by ξ_a the partition into ergodic components of element a and by $\xi(F)$ the measurable hull of F .

The desired result is established by applying Lemmas 6.1, 6.2, 6.3, and 6.4 to $F = W^P$. Note that by the assumption of Theorem 3.1, $\text{Ad}(c)$ is a diagonalizable automorphism of the Lie algebra \mathfrak{g} for all $c \in A$. This implies that the derivative of the element a acts isometrically on W^P .

The general setup for these lemmas is as follows. Let $a \in A$ be a singular element acting isometrically on an invariant foliation F . Denote by $B_1^F(x)$ the closed unit ball about x in the leaf $F(x)$ with respect to the induced metric. Denote by μ_x^F the system of conditional measures on F normalized by the requirement $\mu_x(B_1^F(x)) = 1$ for all x in the support of μ . Denote by I_x the subgroup of all isometries of $F(x)$ which preserve μ_x^F up to a scalar multiple.

The first three lemmas are the analogs of the Lemmas 5.4, 5.5, and 5.6 from [15]; see also Lemmas 3.2, 3.3, and 3.4 in [10].

LEMMA 6.1. [15, Lemma 5.4] *Suppose that a acts isometrically on some $\alpha(a)$ -invariant foliation F which satisfies $\xi_a \leq \xi(F)$. Then for μ -almost every x , I_x is closed and the support of μ_x^F is the orbit of x under the group I_x . Furthermore, $\phi_*\mu_x^F = \mu_{\phi x}^F$ for any $\phi \in I_x$.*

The proof of Lemma 5.4 in [15] (Lemma 3.2 in [10]) works verbatim for the general case.

LEMMA 6.2. *Suppose that an invariant subfoliation F of W_b^- for an element b satisfies the conclusions of Lemma 6.1. Then for μ -almost every x , the support S_x of μ_x^F is a homogeneous submanifold of $F(x)$; more precisely it is a left coset $S_x = L_x x$ of some subgroup L_x .*

Proof. Let E denote the tangent distribution of the foliation F . Let us denote by $\lambda_1, \dots, \lambda_m$ the negative Lyapunov exponents of b and by E^{λ_i} , $i = 1, \dots, m$, the intersections of E with the corresponding Lyapunov subspaces.

The conclusion of Lemma 6.1 states that for μ -almost every x the support S_x of the measure μ_x^F is the orbit of a closed group of isometries. Therefore S_x is a submanifold of the leaf $F(x)$, possibly disconnected. Thus the tangent space $T_x S_x$ depends on x smoothly along the leaf $F(x)$ and measurably on \mathcal{M} . Our main goal it to show that the tangent distribution of S_x is invariant under left translations along S_x in the following sense. If x and hx belong to the same connected component then the left translation by h takes the tangent space at x to the tangent space at hx .

We note that b maps S_x to S_{bx} and contracts F . Using recurrence under the iterates of b and the fact that the tangent vectors in different E^{λ_i} have different exponential rates of contraction one can see that $T_x S_x = \bigoplus_{i=1}^m (T_x S_x \cap E^{\lambda_i})$. Thus to prove that the tangent distribution $T S_x$ is invariant under left translations it suffices to show that $T_x S_x \cap E^{\lambda_i}$ is invariant under left translations for every $i = 1, \dots, m$.

Let us fix i and put $V_x = T_x S_x \cap E^{\lambda_i}$. Note that for $y \in S_x$ we have $S_x = S_y$. Let $\beta_x(y)$ be the maximal angle between the subspaces V_y and $R_g(V_x)$, where $g = x^{-1}y$. Then

$$K(x) = \limsup_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \sup\{\beta_x(y) \mid d(x, y) < \epsilon\}$$

gives a maximal rate of deviation of the distribution V_x from the left invariant distribution and thus plays the role of maximal curvature. Since $V_x = T_x S_x \cap E^{\lambda_i}$, it is measurable in x and smooth along S_x . Thus we see that $\beta_x(y)$ is measurable in x and Lipschitz along S_x . Thus $K(x)$ exists and is measurable. Moreover, it is easy to see that $K(y)$ gives an upper bound for the derivative of β_x at y . Hence to prove that V_x is invariant under left translations for almost every x it suffices to show that $K(x) = 0$ almost everywhere.

To show this we use the element b for which $\lambda(b) < 0$. Thus the iterates of b exponentially contract the leaves of F . We note that b maps V_x to $V_{b^i x}$ and left invariant distributions to left invariant distributions. Since V_x is contained in a single Lyapunov subspace, we see that the exponential contractions in all directions V_x are the same. Since the derivative of b along E^{λ_i} is semisimple, b preserves the angles. Since b also contracts the distances it follows that $K(b^i x)$ goes to infinity unless $K(x) = 0$. Since Poincaré recurrence under b implies that this is impossible, we conclude that $K(x) = 0$ almost everywhere. Hence V_x is left invariant.

Thus we conclude that the tangent distribution of every connected component of S_x is left invariant in the sense that if x and hx belong to the same connected component then the left translation by h takes the tangent space at x to the tangent space at hx . Hence $S_x = \bigcup_{x_i} H_{x_i}$.

Let us now show that the support is connected. Suppose to the contrary that the support is a union $\bigcup A_i$ of at least two affine subspaces A_i . Let d_x denote the minimum of the distances from x to any A_i which does not contain x . Since the support is a closed subset, $d_x > 0$ for all x . Note that $d_{b^n x} \rightarrow 0$ as $n \rightarrow \infty$. This again contradicts Poincaré recurrence under b . \square

Remark. If we do not assume that the derivative of b along E^{λ_i} is semisimple, it may contain Jordan blocks. In this case the angles may not be preserved, but the distortion of the angles is at most polynomial and does not affect the above argument since the distances are contracted exponentially.

LEMMA 6.3. *Suppose that a b -invariant subfoliation $F \subset W_b^-$ satisfies the conclusions of Lemma 6.1 and suppose that the support S_x of μ_x^F is, for μ -almost every x , a left coset $S_x = L_x x$ of some subgroup L_x . Then μ_x^F is a Haar measure on $S_x = L_x x$.*

A modified version (see Lemma 3.2 in [10]) of the proof of Lemma 5.4 in [15] can be applied for the general case. It shows that the isometries of Lemma 6.1 actually preserve the measure. We note that the existence of such a transitive group of measure-preserving isometries implies that the measure is Haar.

LEMMA 6.4. *Suppose that for an α -invariant foliation F the conditional measure μ_x^F is, for μ -almost every x , a Haar measure on a left coset $S_x = L_x x$ of some subgroup L_x . Then $L_x = L$ is constant almost everywhere. Hence F satisfies property (A) and $aL a^{-1} = L$.*

Proof. The proof is the same as the proof in Section 5.2 of the fact that H_x is constant almost everywhere. \square

6.2. *Step 2.* In this step we complete the proof of Theorem 3.1 by showing that the foliation W_b^- satisfies property (A). We will prove this by using the results of Step 1 and establishing property (A) inductively for larger and larger invariant subfoliations of W_b^- .

Let P_i , $1 \leq i \leq l$, be the Lyapunov half-spaces containing b and let E^{P_i} (W^{P_i}) be the corresponding coarse Lyapunov distributions in E_b^- (subfoliations of W_b^-). The results of the first step imply that all foliations W^{P_i} satisfy property (A).

To arrange the inductive process we restrict the action to a generic 2-plane which contains b and intersects all the Lyapunov hyperplanes in generic lines. For each Lyapunov hyperplane ∂P_i we choose a generic singular element c_i in the intersection of ∂P_i with the generic 2-plane. We can now reorder elements c_i and Lyapunov half-spaces P_i in such a way that for any $1 \leq j \leq l - 1$ the distribution $E^{P_1} \oplus \dots \oplus E^{P_j}$ is integrable and is contracted by c_{j+1} .

Now the fact that the foliation W_b^- satisfies property (A) follows from inductive application of the following lemma, which is the main part of Step 2.

LEMMA 6.5. *Let W be an invariant subfoliation of W_c^l and F be an invariant subfoliation of W_c^- for some generic singular element c . Suppose that W and F satisfy property (A) and are jointly integrable to the foliation $W \oplus F \subset W_b^-$ for some element b . Then $W \oplus F$ satisfies property (A).*

Proof. We prove this lemma inductively by adding one-dimensional subfoliations of W to the foliation F until we exhaust the whole of $F \oplus W$. We first set up the inductive process and then use Lemma 6.6 to show that on each step of this process we obtain a foliation that satisfies property (A).

Notation. Throughout the proof we denote by H the one-dimensional subfoliation of W which is being added at the current step of the induction and by D the sum of all one-dimensional subfoliations of W which have already been added (thus D is trivial in the beginning). We denote by E^F , E^W , E^D , and E^H the tangent distributions of foliations F , W , D , and H correspondingly.

Description of the induction. To arrange the induction we will choose a basis of E^W and an ordering of this basis according to which the corresponding one-dimensional subfoliations will be added. Since $W \subset W_b^-$ is an invariant subfoliation, E^W is an invariant subalgebra of E_b^- . Since E_b^- is nilpotent so is E^W . Thus the central series of E^W gives the invariant filtration $E^W = V^k \supset \dots \supset V^1 \supset V^0 = 0$ such that $[V^i, V^i] \subset V^{i-1}$, for $i = 1, \dots, k$. In particular each V^i is a subalgebra invariant under the adjoint action of b and c . Recall that $L(\mu)$ is the Lie algebra of $\Lambda(\mu)$, the subgroup of G whose left translations preserve μ . Hence $V^i \cap L(\mu)$ is a subalgebra of V^i , for $i = 1, \dots, k$, which is invariant since $L(\mu)$ is. To obtain the desired ordered basis of E^W we first choose a basis of $V^1 \cap L(\mu)$ and complement it to a basis of V^1 . After that we complement it to a basis of $V^1 \oplus V^2 \cap L(\mu)$ and then to a basis of $V^1 \oplus V^2$. Continuing in the same way we obtain a basis of E^W with the natural partial order. The order of the basis directions within $V^1 \cap L(\mu)$ and within each step of the complementing can be chosen arbitrarily. We note that the subspace spanned by the first several basis directions is always a subalgebra and thus is integrable. Hence the foliations D and $D \oplus H$ above are well defined.

Since $\text{Ad}(b)$ and $\text{Ad}(c)$ are diagonalizable, commute, and leave all subspaces V^i and $V^i \cap L(\mu)$ invariant, the basis above can be chosen in such a way that on each inductive step the foliations D and $D \oplus H$ are invariant for some multiples sb and tc in A of the elements b and c . Indeed, if all eigenvalues of $\text{Ad}(b)$ and $\text{Ad}(c)$ on E^W are real the basis above can be chosen as a common eigenbasis for $\text{Ad}(b)$ and $\text{Ad}(c)$. In this case on each step the foliations D and $D \oplus H$ are clearly invariant under b and c . If $\text{Ad}(b)$ and $\text{Ad}(c)$ have complex eigenvalues, a similar basis can be chosen using the real normal forms for $\text{Ad}(b)$ and $\text{Ad}(c)$. We then add the one-dimensional foliations in such a way that if one direction of a complex eigenspace gets added, the other gets added in the next step. Now choose multiples sb and tc of b and c in such a way that their eigenvalues corresponding to E^H are real. Then on each inductive step the foliations D and $D \oplus H$ are invariant under sb and tc .

Thus we have arranged the inductive process in such a way that on each step the foliations D and $D \oplus H$ are well defined and invariant for some multiples sb and tc of the elements b and c . We also note that the distributions $E^F \oplus E^D$, $E^F \oplus E^H$ and $E^F \oplus E^D \oplus E^H$ are integrable. This clearly follows from the next observation. If a subdistribution E of E^W is a subalgebra then the direct sum $E \oplus E^F$ is also a subalgebra and thus is integrable. Indeed, since both E and E^F are subalgebras it suffices to show that the Lie bracket of an element in E with an element in E^F belongs to E^F . Since $F \oplus W$ is integrable it must belong to $E^F \oplus E^W$. We note that if the Lie bracket is non-zero then its Lyapunov exponent is the sum of the Lyapunov exponents of the original elements. Since the kernels of these two Lyapunov exponents are different we see that the kernel of their sum is different from both of them. Since $E^W \subset E_c^I$ and all Lyapunov exponents of E_c^I have the same kernel we see that the Lyapunov subspace of the bracket cannot be in E^W . Hence it must belong to E^F .

Inductive step. It follows from the construction of the basis that on each step either E^H is contained in $L(\mu)$ or the intersection of $E^D \oplus E^H$ with $L(\mu)$ is contained in E^D .

If the current H is contained in $L(\mu)$ the inductive step is simple. Indeed, in this case for μ -almost every x the conditional measure on the leaf $F \oplus D \oplus H(x)$ is invariant under the left translations by the Lie subgroup $U \subset G$ corresponding to $(E^{F \oplus D} \cap L(\mu)) \oplus E^H$. Hence this conditional measure is an integral of Haar measures on the left cosets of U . Moreover, we see that it must be supported on a single left coset of U since we know that the conditional measure on the leaf $(F \oplus D)(x)$ is supported on a single left coset of $U \cap (F \oplus D)$. Thus $F \oplus D \oplus H$ satisfies property (A).

The next lemma proves the inductive step in the case when the intersection of $E^D \oplus E^H$ with $L(\mu)$ is contained in E^D . This will complete the proof of Lemma 6.5. \square

LEMMA 6.6. *Let F , D and H be b - and c -invariant subfoliations of W_b^- . Suppose that H is one-dimensional, that the distributions $E^F \oplus E^D \oplus E^H$, $E^F \oplus E^D$, $E^F \oplus E^H$ and $E^D \oplus E^H$ are integrable, and that we have $(D \oplus H) \subset W_c^I$ and $F \subset W_c^-$.*

If the foliations $F \oplus D$ and $D \oplus H$ satisfy property (A) and if the intersection of $E^D \oplus E^H$ with $L(\mu)$ is contained in E^D , then the foliation $F \oplus D \oplus H$ also satisfies property (A).

Proof. We consider a measurable partition, which is subordinate to the foliation $F \oplus D \oplus H$ and tiles the intersection of each leaf with a set of measure at least 0.99 by small ‘boxes’

of roughly the same size. The boxes can be chosen in the following way. Take $y \in \mathcal{M}$ and small open neighborhoods U and V of y in $F(y)$ and $D(y)$ correspondingly. Since $F \oplus D$ is integrable we can define a rectangle R_y as the set of intersections

$$\{z \in (F \oplus D)(y) : z = F(v) \cap D(u), \text{ where } v \in V \text{ and } u \in U\}.$$

Similarly, since $(F \oplus D) \oplus H$ is integrable we can define a box T_y as the product of the above type of R_y with an interval in one-dimensional leaf $H(y)$. For any x in the above set of measure at least 0.99 we denote by $T(x)$ the box it belongs to.

Since the foliation $D \oplus H$ satisfies property (A) there exists a compact set K of measure at least 0.98 such that for any point x the intersection of the box $T(x)$ with $K \cap (D \oplus H)(x)$ is contained in the intersection of $T(x)$ with a single left coset of $\Lambda(\mu) \cap (D \oplus H)$. Since the foliation $F \oplus D$ also satisfies property (A) we may assume the similar relationship between K and $F \oplus D$.

Denote by X the set of points in K whose distance from the union of the boundaries of the boxes is at least γ . By the ‘boundary of a box’ we understand its leafwise boundary with respect to the leaf of $F \oplus D \oplus H$. The measurable partition can be easily chosen in such a way that the total measure of the union of these boundaries is small and $\mu(X) > 0.96$ for some $\gamma > 0$.

Let μ_X be the restriction of the measure μ to the ‘good’ set X , i.e. $\mu_X(\cdot) = \mu(X \cap \cdot)$. We note that the measure μ_X may not be invariant. Let us consider the system of the conditional measures of μ_X with respect to the measurable partition into the boxes described above (the set not covered by the boxes has μ_X measure 0). These measures will be referred to as the conditional measures of boxes.

We try to slice each box $T(x)$ into three parts T_x^L , T_x^M and T_x^R of equal conditional measure by two leaves of the foliation $F \oplus D$, where T_x^M is the middle part. This may not be possible if the intersection of a box with a single leaf of $F \oplus D$ has positive conditional measure, but then we can proceed as in the second case of the dichotomy below. For each $x \in X$ we denote by $d(x)$ the minimal distance (along the one-dimensional H direction) between T_x^L and T_x^R . For the measurable function $d(x)$ we have the following dichotomy:

- (1) either $d(x) \geq d > 0$ on a set Y which consists of whole boxes and has measure $\mu_X(Y) > 0.95$;
- (2) or for every box in a set of positive measure, at least $1/3$ of its conditional measure is concentrated on the intersection with a single leaf of $F \oplus D$.

In the latter case we argue as follows. Consider a point x in this set of positive measure and the corresponding box $T(x)$. Since the foliation $F \oplus D$ satisfies property (A) the conditional measure on the leaf $(F \oplus D)(x)$ is Haar on a single left coset of $\Lambda(\mu) \cap (F \oplus D)$. The fact that the intersection of this left coset with the box $T(x)$ carries at least $1/3$ of the conditional measure of $T(x)$ means that the measure of this intersection dominates the measure of its small tubular neighborhood in the leaf $(F \oplus D \oplus H)(x)$. By applying measure-preserving transformation b^{-1} , which expands $F \oplus D \oplus H$, we see that, in a set of positive measure, the Haar measure on a single left coset of $\Lambda(\mu) \cap (F \oplus D)$ dominates a larger and larger part of the leaf of $F \oplus D \oplus H$. Using this and recurrence under b^{-1} we conclude that the conditional measure on a typical leaf of $F \oplus D \oplus H$ is Haar on a single left coset of $\Lambda(\mu) \cap (F \oplus D)$. This proves that $F \oplus D \oplus H$ satisfies property (A).

To complete the proof it remains for us to eliminate the first alternative of the dichotomy. Now we use the special properties of $D \oplus H$.

Recall that the intersection of the compact set K with any box consists of the pieces of left cosets of $\Lambda(\mu) \cap D$, with at most one piece for each leaf of $D \oplus H$. Hence the projection (along the D direction) of this intersection to the ' $F \oplus H$ face of the box' is the graph of a continuous (not necessarily everywhere defined) function from the F -direction to the H -direction. Since K is compact the family of these functions is equicontinuous. We will show that this equicontinuity contradicts the recurrence under the action of a properly chosen element in A .

Since b contracts $F \oplus D \oplus H$ we can find a sufficiently large number n such that the size of the image of any box under the action of b^n is γ -small. The distance along the H -direction between the images of T_x^L and T_x^R , which was at least d , becomes at least d' . Let us fix some $\epsilon < d'$ and consider $\delta > 0$ given for this ϵ by the equicontinuity. Since c acts isometrically on $D \oplus H$ and contracts F we can choose k so large that the image of any box under $c^k b^n$ is δ -small in the F -direction. We conclude that the element $c^k b^n$ satisfies the following conditions:

- (1) the image of any box is δ -narrow in the F -direction;
- (2) the distance along the H -direction between the images of the right and the left parts of any box is at least ϵ ;
- (3) the diameter of the image of any box is less than γ .

Consider a point $y \in Y$ and the corresponding box T . Since the set X does not intersect the γ -neighborhood of the boundaries of the boxes, (3) implies that $c^k b^n(T) \cap X$ lies in one box. Hence by (1) and (2), $c^k b^n(T^L)$ and $c^k b^n(T^R)$ cannot both intersect X . This implies that there exists a set $Z \subset Y$ with $\mu_X(Z) \geq \frac{1}{3}\mu_X(Y)$ for which $c^k b^n(Z) \cap X = \emptyset$. But this is impossible since $\mu(X) > 0.96$ and $\mu(c^k b^n(Z)) = \mu(Z) \geq \mu_X(Z) \geq \frac{1}{3}\mu_X(Y) > \frac{1}{3}0.95$.

This shows that the second alternative of the dichotomy is impossible and thus proves that foliation $F \oplus D \oplus H$ satisfies property (A). \square

This completes the proof of Theorem 3.1.

7. Proof of Corollary 3.3

We will use the notations from Theorem 3.1 and its proof.

We first claim that the subgroup H is reductive. Without loss of generality we suppose that H itself rather than a conjugate intersects Γ in a lattice. If H is not reductive, then it can be decomposed as $H = SK \text{ Rad}$, where S is semisimple without compact factors, K is a compact semisimple group, and Rad is the radical of H . Then $K \text{ Rad}$ intersects Γ in a lattice (cf. e.g. [31, Corollary 2.12]). That H is reductive follows once we know that the unipotent radical R_u of H is trivial. If not, we may pick a non-trivial one-parameter unipotent subgroup U of R_u . Any non-trivial $u \in U$ is horocyclic in the sense of [28, Definition 2], as follows from the Jacobson–Morozov lemma. Since Γ is a cocompact lattice in G , we may apply [26, Corollary 2] to the subgroup $K \text{ Rad}$. Hence $K \text{ Rad}$ will contain a copy of sl_2 , which is impossible. Thus H is reductive.

Recall that the Haar measure on Hx is ergodic with respect to a subgroup generated by unipotents contained in H . In particular, it is ergodic with respect to S . Note that S acts

trivially on the maximal Euclidean quotient of Hx (the quotient by SK). Hence S cannot act ergodically unless H is semisimple.

Since H is semisimple the connected component of its normalizer is a product of H with the centralizer of H (connected components always). Thus H intersects Γ in a lattice which is Zariski dense in H by the Borel Density Theorem. Hence the centralizer of H is the same as the centralizer of $H \cap \Gamma$. The latter is an intersection of centralizers of all the elements of $H \cap \Gamma$. Since Γ is arithmetic by Margulis' Arithmeticity Theorem, G is defined over a number field F , and Γ is commensurable with the integer points of G with respect to F . Hence the centralizer of $\gamma \in \Gamma \cap H$ is defined over F , and so is $Z_G H$. Hence $\Gamma \cap N_G H$ is commensurable with integer points of $N_G H$ which form a lattice on $N_G H$ by Borel and Harish-Chandra's theorem [1].

8. Proof of Theorem 4.1

In this section we will reduce Theorem 4.1 to Theorem 3.1. As α is given by diagonalizable automorphisms, the homomorphism from \mathbb{Z}^k to the automorphisms of G extends to \mathbb{R}^k . Hence we can form the semidirect product groups $G' = \mathbb{R}^k \ltimes G$, $\Gamma' = \mathbb{Z}^k \ltimes G$ and its compact quotient G'/Γ' . Then the action α' of \mathbb{R}^k on G'/Γ' is the action of \mathbb{R}^k induced from \mathbb{Z}^k . We will freely use all the notation from the main theorem, denoting any relevant object for the induced action by a α' .

Then α' almost satisfies the assumptions of Theorem 3.1. The main assumption on the existence of suitable singular elements is already postulated in the formulation of Theorem 4.1. If the eigenvalues of the automorphisms $\alpha(\mathbb{Z}^k)$ have real eigenvalues so do those of $\alpha(\mathbb{R}^k)$. Thus we only need to discuss what to do when α is weakly mixing. Consider the natural fibration of G'/Γ' over $T = \mathbb{R}^k/\mathbb{Z}^k$. Then any eigenfunction f for \mathbb{R}^k is an eigenfunction for \mathbb{Z}^k on almost every fiber G/Γ . Hence f is constant along almost every fiber. In the proof of the main theorem, we consider the function $x \mapsto H'_x$. If α is weakly mixing, the H'_x are constant along almost every fiber G/Γ , as desired. Since the H'_x are generated by unipotent elements, it is clear that $H'_x \subset G$.

Note that conclusions (1)–(3) of Theorem 3.1 immediately imply the corresponding claims of Theorem 4.1. Furthermore, statements (4) and (5) of Theorem 3.1 imply conclusion (4) of Theorem 4.1 if we can show that the normalizer $N_{G'}(H')$ intersects Γ' in a lattice. This follows from the following result. We thank D. Witte for providing a proof for us. As we could not find precise references for standard facts, we give a detailed account.

PROPOSITION 8.1. *Suppose:*

- G is a 1-connected, nilpotent Lie group;
- A is a 1-connected, Abelian Lie group which acts on G by automorphisms;
- $G' = A \ltimes G$;
- H is a closed, connected subgroup of G ;
- Γ is a lattice in G ;
- $\Gamma \cap H$ is a lattice in H ;
- \mathbb{Z}^k is a lattice in A .

Let $\Gamma' = \mathbb{Z}^k \rtimes \Gamma$. Then $\Gamma' \cap N_{G'}(H)$ is a lattice in $N_G(H)$, and also $\Gamma \cap N_G(H)$ is a lattice in G .

Proof. Since Γ' is a polycyclic group, Γ' admits a faithful representation ρ into $\mathrm{GL}(n, \mathbb{Z})$ by [25, Theorem 4.12] such that the image of the nilradical of Γ' consists of unipotent matrices. Thus $\rho(\Gamma')$ is unipotent. As Γ is nilpotent, ρ extends to a representation of G into a unipotent subgroup of $\mathrm{GL}(n, \mathbb{R})$ by [25, Theorem 2.11]. Also $\rho|_{\mathbb{Z}^k}$ extends to a representation of A into $\mathrm{GL}(n, \mathbb{R})$ such that $\rho(A)$ normalizes $\rho(G)$.

Hence we may assume $G' \subset \mathrm{GL}(\ell, \mathbb{R})$, for some ℓ , in such a way that:

- G is unipotent; and
- $G' \cap \mathrm{GL}(\ell, \mathbb{Z})$ is (commensurable with) Γ .

Let:

- L be the Zariski closure of G' in $\mathrm{GL}(\ell, \mathbb{R})$; and
- $\Lambda = L \cap \mathrm{GL}(\ell, \mathbb{Z})$.

Dani's generalization of the Borel Density Theorem [3] shows that Λ is Zariski dense in a cocompact subgroup of L , which implies that L has no characters defined over \mathbb{Q} . By the compactness criterion for lattices [25, Theorem 10.19], Λ is a cocompact lattice in L .

Because $\Gamma \cap H$ is Zariski dense in H , we know that H is defined over \mathbb{Q} . Hence $N_L(H)$ is also defined over \mathbb{Q} . Therefore, $\Lambda \cap N_L(H)$ is a cocompact lattice in $N_L(H)$. By [25, Lemma 1.19], we conclude that $\Lambda \cap G' \cap N_L(H)$ is a lattice in $G' \cap N_L(H) = N_{G'}(H)$. This proves the first claim since $\Lambda \cap G'$ is commensurable with Γ' . Finally, that $\Gamma \cap N_G(H)$ is a lattice in G follows immediately since $N_G(H)/\Gamma \cap N_G(H)$ is the fiber of the fibration $N_{G'}(H)/\Gamma \cap N_{G'}(H)$ over the torus $\mathbb{R}^k/\mathbb{Z}^k$. \square

9. Proofs of Theorems 3.6 and 4.5

In this section we prove Theorems 3.6 and 4.5. The proofs are very similar, so we first prove Theorem 3.6 and then in §9.2 describe the changes needed for the nilmanifold case.

9.1. *Proof of Theorem 3.6.* We introduce the following notations: $\alpha = \alpha_1 \times \alpha_2$, $G = G_1 \times G_2$, $\Gamma = \Gamma_1 \times \Gamma_2 \subset G$, and $A = \rho(\mathbb{R}^k) = \{(\rho_1(a), \rho_2(a)) \in G : a \in \mathbb{R}^k\}$, where $\rho = \rho_1 \times \rho_2$ is the diagonal embedding of \mathbb{R}^k into $G = G_1 \times G_2$.

Since (α, μ) satisfies the assumptions of Corollary 3.3 there exists a subgroup $H \subset \Lambda(\mu) \subset G$ such that μ is supported on a compact coset $N_G(H)y \subset G/\Gamma$ of the normalizer of H in G . By passing if necessary to the algebraically isomorphic action of $y^{-1}Ay$ on $(y^{-1}Hy) \setminus y^{-1}N_G(H)y/\Gamma$, we may assume that $\bar{e} = \{e\Gamma\} \in G/\Gamma$ is a density point for μ , and hence μ is supported on $N_G(H)\bar{e} \subset G/\Gamma$.

Let $H_i = H \cap G_i$, $i = 1, 2$. We would like to factorize by $H_1 \times H_2 \subset H$, so we first prove that $H_1 \times H_2$ is normal in G . By symmetry, it suffices to show that H_1 is normal in G_1 . For this we note that since μ is supported on $N_G(H)$ and projects to the Haar measure on the G_1/Γ_1 , the projection of $N_G(H)$ to G_1 must be onto. Hence for any $g_1 \in G_1$ there exists $g_2 \in G_2$ such that $(g_1, g_2) \in N_G(H)$, i.e. $(g_1^{-1}, g_2^{-1})H(g_1, g_2) = H$. This, together with the equality $(g_1^{-1}, g_2^{-1})G_1(g_1, g_2) = G_1$, shows that $(g_1^{-1}, g_2^{-1})(G_1 \cap H)(g_1, g_2) = (G_1 \cap H)$. But the latter implies that $g_1^{-1}H_1g_1 = H_1$. Thus we have proved that $H_1 \times H_2$ is normal in G .

Hence we can consider the factor action α' on G'/Γ' , where $G' = (H_1 \times H_2) \setminus G = G'_1 \times G'_2$ with $G'_i = H_i \setminus G_i$, $i = 1, 2$. We note that since $\Gamma = \Gamma_1 \times \Gamma_2$, the lattice Γ' in the factor is also the product of lattices Γ'_1 and Γ'_2 , which are the projections of Γ_1 and Γ_2 to G'_1 and G'_2 respectively. Recall that for the measure μ we have a natural measurable partition ξ into compact cosets Hx , $x \in N_G(H)$, with Haar conditional measures. It follows that the factor measure μ' on G'/Γ' is invariant under the left action of $H' = (H_1 \times H_2) \setminus H$, and that the projection of ξ to G'/Γ' is a measurable partition ξ' into compact cosets $H'x$. We note that since $H' \cap G'_i = \{e\}$, the subgroup H' projects injectively to G'_1 and G'_2 . Let $H'_1 = p_1(H')$, where p_1 is the projection to the first factor. Since $p_1(H'x) = H'_1 \cdot p_1(x)$, it is easy to see that ξ' projects to a measurable partition ξ'_1 of $\mathcal{M}'_1 = G'_1/\Gamma'_1$ which consists of compact cosets H'_1y . We will show below that ξ'_1 is coarser than the Pinsker partition for $\alpha'_1(a)$, where α'_1 is the action on \mathcal{M}'_1 and $a \in \mathbb{R}^k$ is from the statement of Theorem 3.6. Since $\alpha'_1(a)$ is a factor of $\alpha_1(a)$ and the latter is, by assumption, a K-automorphism with respect to the Haar measure on G_1/Γ_1 , we conclude that ξ'_1 has to be the trivial partition. Therefore, $\mu'_1 = (p_1)_*\mu'$ is concentrated on a single compact coset H'_1x . Since μ'_1 is Haar, as the projection to \mathcal{M}'_1 of the Haar measure λ_1 on G_1/Γ_1 , we conclude that $H'_1 = G'_1$. Thus the subgroup H' projects injectively and onto G'_1 . By the same argument, H' projects injectively and onto G'_2 . Hence H' is the graph of the isomorphism $f : G'_1 \rightarrow G'_2$ determined by the equality $H' = \{(g, f(g)) \in G' : g \in G'_1\}$.

We know that μ' is invariant under H' and is supported on the normalizer $N_{G'}H'$ of H' in G' . We now describe $N_{G'}H'$ to show that μ' is supported on a single coset $H'x$. If $(g_1, g_2) \in N_{G'}H'$, then for any $g \in G'_1$ we have $(g, f(g)) \in H'$ and thus $(g_1^{-1}gg_1, g_2^{-1}f(g)g_2)$ is again in $N_{G'}H'$. Hence $f(g_1^{-1}gg_1) = g_2^{-1}f(g)g_2$, which implies $f(g) = (f(g_1)g_2^{-1})^{-1}f(g)(f(g_1)g_2^{-1})$. Since this holds for all $g \in G_1$, $z = (f(g_1)g_2^{-1})$ lies in the centralizer of $f(G'_1)$, which coincides with the center $Z(G'_2)$ of G'_2 as f is an isomorphism. Since $z \cdot (g_1, g_2) = (g_1, f(g_1)) \in H'$ we conclude that $N_{G'}H' = Z(G_2) \cdot H'$. Since G'_2 is semisimple, $Z(G'_2)$ is at most countable and thus $N_{G'}H'$ consists of at most countably many cosets of H' . Since μ' is an ergodic measure for the \mathbb{R}^k -action α' supported on $N_{G'}H'$, it is clear that μ' is supported on a single coset $H'x \subset G'/\Gamma'$. Since $\bar{e} \in G'/\Gamma'$ is a density point of μ' , we see that μ' is Haar on $H'\bar{e}$. This implies that μ is Haar on $H\bar{e}$, i.e. μ is algebraic.

Since μ' is Haar on $H'\bar{e}$, the latter is also invariant under α' . Since we know that H' is isomorphic to G'_1 and G'_2 it follows that $H'\bar{e}$ projects onto \mathcal{M}'_1 and $\mathcal{M}'_2 = G'_2/(\Gamma' \cap G'_2)$. In contrast to the case of isomorphism rigidity $H'x$ may not project injectively to \mathcal{M}'_1 and \mathcal{M}'_2 . Thus it may not produce an algebraic isomorphism between the actions α'_1 and α'_2 . However, $H'x$ is compact and thus $H' \cap \Gamma'$ is a uniform lattice in H' . Hence the projection of $H' \cap \Gamma'$ to G'_1 is also a uniform lattice which is contained in Γ'_1 since $\Gamma' = \Gamma'_1 \times \Gamma'_2$, and thus is of finite index in Γ'_1 . Similarly, the projection of $H' \cap \Gamma'$ to G'_2 is of finite index in Γ'_2 . This implies that the invariant coset $H'\bar{e}$ is a graph of an algebraic isomorphism between the actions α'_1 and α'_2 on the finite coverings of G'_1/Γ'_1 and G'_2/Γ'_2 respectively.

To complete the proof it remains for us to show that the measurable partition ξ'_1 of \mathcal{M}'_1 is coarser than the Pinsker partition for $\alpha'_1(a)$. We note that the Pinsker partition for $\alpha'_1(a)$ coincides with the measurable hull of the unstable foliation for $\alpha'_1(a)$ [19, Theorem B].

Thus it suffices to prove that the projection $p_1(\mathfrak{h}')$ of the Lie algebra \mathfrak{h}' of the group H' to the Lie algebra \mathfrak{g}'_1 of the group G'_1 contains the Lie algebra of the unstable distribution of $\alpha'_1(a)$. Consider a Lyapunov exponent χ for the action α' such that $\chi(a) > 0$ and denote by V^χ the corresponding Lyapunov subspace in the Lie algebra \mathfrak{g}' of the group G' . Then χ is also a Lyapunov exponent for at least one of the factors, and V^χ is the direct sum of the corresponding Lyapunov subspaces $V_1^\chi \subset \mathfrak{g}'_1$ and $V_2^\chi \subset \mathfrak{g}'_2$, one of which may be trivial. We recall that both projections p_1 and p_2 are injective on H' . Hence

$$\dim(\mathfrak{h}' \cap V^\chi) \leq \dim V_i^\chi, \quad i = 1, 2. \quad (3)$$

Since μ'_1 is the Haar measure on \mathcal{M}'_1 , the entropy of $\alpha'_1(a)$ is

$$h(\alpha'_1(a)) = \sum \chi(a) \dim(V_1^\chi), \quad (4)$$

where the sum is taken over all Lyapunov exponents χ positive on a . Since for μ -almost every x the conditional measure of μ on the leaf $W_{\alpha(a)}^+(x)$ is Haar on a single left coset Hx , it follows that for μ' -almost every x the conditional measure of μ' on the leaf $W_{\alpha'(a)}^+(x)$ is Haar on a single left coset $H'x$. This implies [20] that the entropy of $\alpha'(a)$ is

$$h(\alpha'(a)) = \sum \chi(a) \dim(H \cap V^\chi), \quad (5)$$

where the sum again is taken over all Lyapunov exponents positive on a . Since $\alpha'_1(a)$ is a factor of $\alpha'(a)$, $h(\alpha'_1(a)) \leq h(\alpha'(a))$. Thus (3), (4) and (5) imply that $\dim(\mathfrak{h}' \cap V^\chi) = \dim V_1^\chi$ for all Lyapunov exponents χ positive on a . Since p_1 is injective on $\mathfrak{h}' \cap V^\chi$, this shows that it is also surjective. Hence $p_1(\mathfrak{h}')$ contains the Lyapunov subspaces of all Lyapunov exponents positive on a and thus contains the unstable distribution $E_{\alpha'_1(a)}^+$. This completes the proof of Theorem 3.6. \square

9.2. *Proof of Theorem 4.5.* In this section we show how to adapt the proof of Theorem 3.6 above to the case of \mathbb{Z}^k -actions by automorphisms of a nilmanifold, and thus establish Theorem 4.5.

Let $\alpha = \alpha_1 \times \alpha_2$ be the diagonal action of \mathbb{Z}^k on $G_1/\Gamma_1 \times G_2/\Gamma_2$, i.e. $\alpha(a)(x_1, x_2) = (\alpha_1(a)x_1, \alpha_2(a)x_2)$. Then the joining (α, μ) satisfies the assumptions of Theorem 4.1. Hence there exists an α -invariant subgroup $H \subset \Lambda(\mu) \subset G_1 \times G_2$ such that μ is supported on a compact coset $N(H)y$ of the normalizer of H in $G_1 \times G_2$. Again by passing if necessary to the algebraically isomorphic action of \mathbb{Z}^k on $(y^{-1}Hy) \setminus y^{-1}N_G(H)y/\Gamma$, we may assume that μ is supported on $N(H)\bar{e}$, where \bar{e} is the projection of the unit to $G_1/\Gamma_1 \times G_2/\Gamma_2$. We again can factorize by $H_1 \times H_2 \subset H$, where $H_i = H \cap G_i$, $i = 1, 2$. In the same way we obtain that the measure μ' in the factor is Haar on the compact cosets of $H' = (H_1 \times H_2) \setminus H$, and that H' projects isomorphically to the factors G'_1 and G'_2 of G_1 and G_2 respectively. We recall that the ergodicity of the element $\alpha_1(a)$ from the statement of Theorem 4.5 implies that it is a K-automorphism [24].

In contrast to the semisimple case above, in this case μ' may not be supported on the single coset, so that μ' and μ may not be algebraic. According to Theorem 4.1 the measures μ' and μ are extensions of a zero-entropy measure for a measurable factor of α with Haar conditional measures in the fibers. This measurable factor can be viewed as an algebraic

factor of the restriction of α to $N(H)\bar{e}$. Even though μ' may not be supported on a single coset of H' , we already know that H and hence H' are α -invariant, and the coset $H'\bar{e}$ is compact. As in the proof above, these facts easily imply that $H'\bar{e}$ is a graph of an algebraic isomorphism between the factor actions on the finite coverings of G'_1/Γ'_1 and G'_2/Γ'_2 , where Γ'_i is the projection of the lattice Γ_i to G'_i , $i = 1, 2$. The measurable factor above can be viewed as a common measurable factor of these factor actions. \square

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