HOLONOMIES AND COHOMOLOGY FOR COCYCLES OVER PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

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ABSTRACT. We consider group-valued cocycles over a partially hyperbolic diffeomorphism which is accessible volume-preserving and center bunched. We study cocycles with values in the group of invertible continuous linear operators on a Banach space. We describe properties of holonomies for fiber bunched cocycles and establish their Hölder regularity. We also study cohomology of cocycles and its connection with holonomies. We obtain a result on regularity of a measurable conjugacy, as well as a necessary and sufficient condition for existence of a contionuous conjugacy between two cocycles.

1. INTRODUCTION

Cocycles and their cohomology play an important role in dynamics. For example, they appear in the study of time changes for flows and group actions, existence and smoothness of absolutely continuous invariant measures, existence and smoothness of conjugacies between dynamical systems, rigidity in dynamical systems and group actions. In this paper we consider cohomology of group-valued cocycles over partially hyperbolic diffeomorphisms.

Definition 1.1. *Let f be a diffeomorphism of a compact manifold $\mathcal{M}$, let G be a topological group equipped with a complete metric, and let $A : \mathcal{M} \to G$ be a continuous function. The G-valued cocycle over f generated by A is the map*

$$\mathcal{A} : \mathcal{M} \times \mathbb{Z} \to G \qquad defined\ by$$

$$\mathcal{A}(x,0) = \mathcal{A}_x^0 = e_G, \quad \mathcal{A}(x,n) = \mathcal{A}_x^n = A(f^{n-1}x) \circ \cdots \circ A(x) \qquad and$$

$$\mathcal{A}(x,-n) = \mathcal{A}_x^{-n} = (\mathcal{A}_{f^{-n}x}^n)^{-1} = (A(f^{-n}x))^{-1} \circ \cdots \circ (A(f^{-1}x))^{-1}, \quad n \in \mathbb{N}.$$

If the tangent bundle of $\mathcal{M}$ is trivial, $T\mathcal{M} = \mathcal{M} \times \mathbb{R}^d$, then the differential Df can be viewed as a $GL(d, \mathbb{R})$-valued cocycle: $A_x = Df_x$ and $\mathcal{A}_x^n = Df_x^n$. More generally, one can consider the restriction of Df to a continuous invariant sub-bundle of $T\mathcal{M}$, for example stable, unstable, or center. In this paper we consider a more general setting of cocycles with values in the group of invertible operators on a Banach space.

A natural equivalence relation for cocycles is defined as follows.

Definition 1.2. *Cocycles* $\mathcal{A}$ *and* $\mathcal{B}$ *are (measurably, continuously) cohomologous if there exists a (measurable, continuous) function* $C : \mathcal{M} \to G$ *such that*

$$(1.1) \qquad \mathcal{A}_x = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1} \;\; \text{for all } x \in \mathcal{M},$$

equivalently, $\mathcal{A}_x^n = C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1}$ *for all* $n \in \mathbb{Z}$ *and* $x \in \mathcal{M}$.

We refer to C as a *conjugacy* between $\mathcal{A}$ and $\mathcal{B}$. It is also called a transfer map. For the differential example above, $C(x)$ can be viewed as a coordinate change on $T_x \mathcal{M}$.

In the context of cocycles over partially hyperbolic systems, two main cohomology problems have been considered so far. One is finding sufficient conditions for existence of a continuous conjugacy. The other is determining whether a measurable conjugacy between two cocycles is necessarily continuous or more regular.

For Hölder continuous real-valued cocycles over systems with local accessibility, the first problem was resolved in [KK], where conditions for existence of a conjugacy were established in terms of su-cycle functionals. Recently, the study of real-valued cocycles was advanced by A. Wilkinson in [W], where she weakened the assumption from local accessibility to accessibility and obtained a positive solution for the second problem. Previous results in this direction were established in [D] for smooth real-valued cocycles over systems with rapid mixing.

For cocycles with values in non-commutative groups, studying cohomology is more difficult. In all results so far, the cocycles satisfied additional assumptions related to their growth, for example fiber bunching for linear cocycles. This property means that noncoformality of the cocycle is dominated by the contraction/expansion of f in the stable/unstable directions. Also, some conclusions in the non-commutative case are different from those in the commutative case. For example, a measurable conjugacy between two cocycles is not necessarily continuous, even when both cocycles are fiber bunched [PW]. Theorem 4.2 gives the first result on continuity of a measurable conjugacy for non-commutative cocycles over partially hyperbolic systems. We make an additional assumption that one of the cocycles is uniformly quasiconformal. The assumption is close to optimal and the theorem extends all similar results for cocycles over hyperbolic diffeomorphisms [Sch, NP, PW, S].

We also obtain a necessary and sufficient condition for existence of a continuous conjugacy between two cocycles in terms of their su-cycle weights. Previously, a sufficient condition was obtained in [KN] for conjugacy to a constant cocycle over a system with local accessibility. However, for non-commutative cocycles the general problem cannot be reduced to the case when one cocycle is constant. We note that in all our results partial hyperbolicity of the base system is pointwise and accessibility is not assumed to be local. The fiber bunching for cocycles is assumed in pointwise sense so, in particular, the results apply to the derivative cocycle along the center direction of a strongly center bunched partially hyperbolic diffeomorphism.

Fiber bunching of a cocycle implies existence of so called stable and unstable holonomies. Some of our results make a weaker assumption of existence of holonomies

in place of fiber bunching. Holonomies are an important and convenient tool in the study of cocycles. In Theorem 3.5 we establish Hölder continuity of holonomies, which is a result of independent interest. We also obtain results on the relationship between conjugacy and holonomies of cocycles, which turns out to be more complicated then in the commutative case. For example, su-cycle weights may be non-trivial for a cocycle continuously cohomologous to a constant one.

In Section 2 we give definitions of partially hyperbolic diffeomorphisms and Banach cocycles. In Section 3 we discuss holonomies and state our result on their regularity. In Section 4 we formulate our results on cohomology of cocycles, and in the last section we give proofs of all the results.

2. Preliminaries

2.1. **Partially hyperbolic diffeomorphisms.** (See [BW] for more details.)

Let $\mathcal{M}$ be a compact connected smooth manifold. A diffeomorphism f of $\mathcal{M}$ is said to be *partially hyperbolic* if there exist a nontrivial Df-invariant splitting of the tangent bundle $T\mathcal{M} = E^s \oplus E^c \oplus E^u$, and a Riemannian metric on $\mathcal{M}$ for which one can choose continuous positive functions $\nu < 1$, $\hat{\nu} < 1$, γ, $\hat{\gamma}$ such that for any $x \in \mathcal{M}$ and unit vectors $\mathbf{v}^s \in E^s(x)$, $\mathbf{v}^c \in E^c(x)$, and $\mathbf{v}^u \in E^u(x)$

$$(2.1) \quad \|Df_x(\mathbf{v}^s)\| < \nu(x) < \gamma(x) < \|Df_x(\mathbf{v}^c)\| < \hat{\gamma}(x)^{-1} < \hat{\nu}(x)^{-1} < \|Df_x(\mathbf{v}^u)\|.$$

We also choose continuous functions μ and $\hat{\mu}$ such that for all x in $\mathcal{M}$

$$(2.2) \quad \mu(x) < \|Df_x(\mathbf{v}^s)\| \text{ if } \mathbf{v}^x \in E^s(x) \quad \text{and} \quad \|Df_x(\mathbf{v}^u)\| < \hat{\mu}(x)^{-1} \text{ if } \mathbf{v}^u \in E^u(x).$$

The sub-bundles E^s, E^u, and E^c are called, respectively, stable, unstable, and center. E^s and E^u are tangent to the stable and unstable foliations W^s and W^u respectively.

An *su-path* in $\mathcal{M}$ is a concatenation of finitely many subpaths which lie entirely in a single leaf of W^s or W^u. A partially hyperbolic diffeomorphism f is called *accessible* if any two points in $\mathcal{M}$ can be connected by an *su*-path.

We say that f is *volume-preserving* if it has an invariant probability measure m in the measure class of a volume induced by a Riemannian metric. It is conjectured that any essentially accessible f is ergodic with respect to such m. The conjecture was proved in cite [BW] under the assumption that f is C^2 and center bunched, or that f is $C^{1+\epsilon}$, $0 < \epsilon < 1$, and strongly center bunched. The diffeomorphism f is called *center bunched* if the functions $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ can be chosen to satisfy

$$(2.3) \qquad\qquad \nu < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma\hat{\gamma}.$$

A $C^{1+\epsilon}$ diffeomorphism f is called *strongly center bunched* if

$$(2.4) \qquad\qquad \nu^\theta < \gamma\hat{\gamma} \quad \text{and} \quad \hat{\nu}^\theta < \gamma\hat{\gamma},$$

for some $\theta \in (0, \epsilon)$ satisfying the inequalities $\nu\gamma^{-1} < \mu^\theta$ and $\hat{\nu}\hat{\gamma}^{-1} < \hat{\mu}^\theta$. These inequalities imply that E^c is θ-Hölder. Note that $(\gamma\hat{\gamma})^{-1}$ is an estimate of non-conformality of $Df|_{E^c}$.

2.2. **Banach cocycles.** Let V be a Banach space, i.e. a vector space equipped with a norm $\|.\|$ such that V is complete with respect to the induced metric. We denote by $L(V)$ the space of continuous linear operators from V to itself. Then $L(V)$ becomes a Banach space when equipped with the operator norm

$$\|A\| = \sup\{\|Av\| : v \in V, \|v\| \le 1\}, \quad A \in L(V).$$

We denote by $GL(V)$ the set of invertible elements in $L(V)$. The set $GL(V)$ is an open subset of $L(V)$ and a group with respect to composition. We use the following metric on $GL(V)$, with respect to which it is complete,

$$(2.5) \qquad d(A,B) = \text{dist}_{GL(V)}(A,B) = \|A - B\| + \|A^{-1} - B^{-1}\|.$$

We call a $GL(V)$-valued cocycle $\mathcal{A}$ a *Banach cocycle*. It is called β-*Hölder* if its generator $A : \mathcal{M} \to GL(V)$ is Hölder continuous with exponent β with respect to the metric d. We note that on any compact set $S \subset GL(V)$ the distance $d(A,B)$ is Lipschitz equivalent to $\|A - B\|$ by Lemma 2.1 below. Therefore, since $\mathcal{M}$ is compact, a cocycle $\mathcal{A}$ is β-Hölder if and only if

$$\|A(x) - A(y)\| \le c \, \text{dist}(x,y)^\beta \quad \text{for all } x, y \in \mathcal{M}.$$

Lemma 2.1. *Suppose that for a subset $S \in GL(V)$ there exists M such that $\|A\| \le M$ and $\|A^{-1}\| \le M$ for all $A \in S$. Then for all $A, B \in S$ we have*

$$M^{-1}\|A^{-1}B - Id\| \le \|A - B\| \le d(A,B) = d(A^{-1}, B^{-1}) \le$$
$$\le (M^2 + 1)\|A - B\| \le M(M^2 + 1)\|A^{-1}B - Id\|.$$

Proof. The equality is clear from the definition of d, and the next inequality follows from the estimate

$$\|A^{-1} - B^{-1}\| \le \|A^{-1}\| \cdot \|B - A\| \cdot \|B^{-1}\| \le M^2\|A - B\|.$$

The other inequalities are obtained similarly. $\qquad\qquad\square$

Definition 2.2. *A cocycle $\mathcal{A}$ over f is called β fiber bunched if it is β-Hölder and*

$$(2.6) \qquad \|\mathcal{A}(x)\| \cdot \|\mathcal{A}(x)^{-1}\| \cdot \nu(x)^\beta < 1 \quad and \quad \|\mathcal{A}(x)\| \cdot \|\mathcal{A}(x)^{-1}\| \cdot \hat{\nu}(x)^\beta < 1,$$

for all x in $\mathcal{M}$, where ν and $\hat{\nu}$ are as in (2.1).

This means that nonconformality of $\mathcal{A}$ is dominated by the expansion/contractions along unstable/stable foliations in the base. Note that the cocycle $Df|_{E^c}$ for a strongly center bunched, (2.4), partially hyperbolic diffeomorphism is θ fiber bunched.

We can view the generator A as the automorphism of the trivial vector bundle $\mathcal{V} = \mathcal{M} \times V$ given by $A(x,v) = (fx, A(x)v)$, and $\mathcal{A}_x^n$ as a linear map between the fibers $\mathcal{V}_x$ and $\mathcal{V}_{f^n x}$. We deal with the case of a trivial bundle for convenience. Our results extend directly to linear cocycles defined more generally as bundle automorphisms, see [KS] for a description of this setting.

2.3. **Standing assumptions.** *In this paper,*
- $\mathcal{M}$ *is a compact connected smooth manifold;*
- *f is an accessible partially hyperbolic diffeomorphism of $\mathcal{M}$ that preserves a volume m and is either C^2 and center bunched, or $C^{1+\epsilon}$ and strongly center bunched;*
- *$\mathcal{A}$ and $\mathcal{B}$ are $GL(V)$-valued continuous cocycles over f, where V is a Banach space.*

3. Holonomies and their regularity

An important role in the study of cocycles is played by holonomies. They were introduced by M. Viana in [V] for linear cocycles and further developed and used in [ASV, KS]. For a fiber bunched linear cocycle $\mathcal{A}$, a holonomy can be obtained as a limit of the products $(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$. Convergence and limits of such products have been studied for various types of group-valued cocycles whose growth is slower than the contraction/expansion in the base (see e.g. [NT, PW, dlLW]). It is related to existence of strong stable/unstable manifolds for the extended system on the bundle. We use the axiomatic definition of holonomies given in [V, ASV]. We note, however, that the resulting object is, in general, non-unique, see discussion after Corollary 4.9.

Definition 3.1. *A stable holonomy for a cocycle $\mathcal{A}$ is a continuous map $H^{\mathcal{A},s} : (x,y) \mapsto H_{x,y}^{\mathcal{A},s}$, where $x \in \mathcal{M}$ and $y \in W^s(x)$, such that*

- (H1) *$H_{x,y}^{\mathcal{A},s}$ is an element of $GL(V)$, viewed as a map from $\mathcal{V}_x$ to $\mathcal{V}_y$;*
- (H2) *$H_{x,x}^{\mathcal{A},s} = \mathrm{Id}$ for every $x \in \mathcal{M}$ and $H_{y,z}^{\mathcal{A},s} \circ H_{x,y}^{\mathcal{A},s} = H_{x,z}^{\mathcal{A},s}$;*
- (H3) *$H_{x,y}^{\mathcal{A},s} = (\mathcal{A}_y^n)^{-1} \circ H_{f^n x, f^n y}^{\mathcal{A},s} \circ \mathcal{A}_x^n$ for all $n \in \mathbb{N}$.*

We say that a stable holonomy is β-Hölder (along the leaves of W^s) if it satisfies the following additional property: for any $R > 0$ there exists K such that

- (H4) $\|H_{x,y}^{\mathcal{A},s} - \mathrm{Id}\| \le K \, \mathrm{dist}_{W^s}(x,y)^\beta$ *for any $x \in \mathcal{M}$ and $y \in W_R^s(x)$.*

Here dist_{W^s} denotes the distance along a leaf of the stable foliation W^s, and $W_R^s(x)$ denotes the ball in $W^s(x)$ centered at x of radius R in this distance. By Lemma 2.1, the left hand side of (H4) is equivalent to the $GL(V)$ distance $d\,(H_{x,y}^{\mathcal{A},s}, \mathrm{Id})$ on the compact set $\{H_{x,y}^{\mathcal{A},s} : \; x \in \mathcal{M}, \; y \in \overline{W_R^s(x)}\,\}$.

Fiber bunched cocycles have a canonical holonomy. The following result was proved for finite dimensional Banach spaces V, but the arguments work for the general case without any modifications.

Proposition 3.2 (Proposition 4.2 [KS], cf. Proposition 3.4 [ASV])**.**
Suppose that a cocycle $\mathcal{A}$ is β fiber bunched. Then for any $x \in \mathcal{M}$ and $y \in W^s(x)$,

$$(3.1) \qquad\qquad H_{x,y}^{\mathcal{A},s} \stackrel{def}{=} \lim_{n\to\infty} (\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n$$

exists and satisfies (H1,2,3,4). The stable holonomy for $\mathcal{A}$ satisfying (H4) is unique.

Remark 3.3. *This proposition holds under a slightly weaker fiber bunching assumption* [S, Proposition 4.4]: *there exist $\theta < 1$ and L such that for all $x \in \mathcal{M}$, $n \in \mathbb{N}$,*

$$(3.2) \quad \|\mathcal{A}_x^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| \cdot (\nu_x^n)^\beta < L\,\theta^n \quad and \quad \|\mathcal{A}_x^{-n}\| \cdot \|(\mathcal{A}_x^{-n})^{-1}\| \cdot (\hat{\nu}_x^{-n})^\beta < L\,\theta^n,$$

where ν_x^n, $\hat{\nu}_x^n$ are defined as in (5.3). *In fact, all results in this paper hold under this version of fiber bunching assumption.*

Definition 3.4. *A stable holonomy for a cocycle $\mathcal{A}$ satisfying* (3.1) *is called* standard.

By definition, the standard stable holonomy of $\mathcal{A}$ is unique, if it exists. By the proposition, the only β-Hölder stable holonomy for a β fiber bunched cocycle is the standard one. However, there are non-standard stable holonomies of lower regularity even for a constant fiber bunched cocycle over an Anosov automorphism.

We use similar definitions for an unstable holonomy $H^{\mathcal{A},u}$. As in Proposition 3.2, any β fiber bunched cocycle $\mathcal{A}$ has the standard unstable holonomy obtained as

$$(3.3) \quad H_{x,y}^{\mathcal{A},u} = \lim_{n\to\infty}\left((\mathcal{A}_y^{-n})^{-1} \circ (\mathcal{A}_x^{-n})\right) = \lim_{n\to\infty}\left(\mathcal{A}_{f^{-n}y}^n \circ (\mathcal{A}_{f^{-n}x}^n)^{-1}\right), \quad y \in W^u(x),$$

It satisfies (H1,2,4,) above with $y \in W^u(x)$ and

(H3$'$) $H_{x,y}^{\mathcal{A},u} = (\mathcal{A}_y^{-n})^{-1} \circ H_{f^{-n}x,\,f^{-n}y}^{\mathcal{A},u} \circ \mathcal{A}_x^{-n}$ for all $n \in \mathbb{N}$.

We establish global Hölder continuity of the stable holonomy for fiber bunched cocycles. A similar result holds for the unstable holonomy.

Theorem 3.5. *Suppose that a cocycle $\mathcal{A}$ is β fiber bunched. Then there exists α, $0 < \alpha < \beta$, such that the standard holonomy $H^{\mathcal{A},s}$ as in* (3.1) *is globally α-Hölder in the following sense. For any $R > 0$ there exist $\delta > 0$ and $C > 0$ so that*

If $y \in W_R^s(x)$, $y'' \in W_R^s(x'')$, $dist(x,x'') < \delta$ and $dist(y,y'') < \delta$, then
$$d(H_{x,y}^s, H_{x'',y''}^s) \le C \max\left\{dist(x,x'')^\alpha, dist(y,y'')^\alpha\right\}.$$

The choice of the Hölder exponent α is explicit and is described in the beginning of the proof. It depends on the system in the base and on the "relative degree of non-conformality" of the cocycle $\mathcal{A}$.

In the absence of fiber bunching, natural examples of cocycles with standard holonomies are given by small perturbation of a constant $GL(d,\mathbb{R})$-valued cocycle.

Proposition 3.6. *Let $\mathcal{A}$ be a constant $GL(d,\mathbb{R})$-valued cocycle generated by A. If $B : \mathcal{M} \to GL(d,\mathbb{R})$ is Hölder continuous and is sufficiently C^0 close to A, then the cocycle generated by B has Hölder continuous standard holonomies.*

4. Cohomology of cocycles

First we consider the question whether a measurable conjugacy between two cocycles is continuous. For non-commutative cocycles, the answer is not always positive, even when both cocycles are fiber bunched. Indeed, in [PW, Section 9], M. Pollicott

and C. P. Walkden constructed an example of two smooth $GL(2, \mathbb{R})$-valued cocycles over an Anosov toral automorphism that are measurably (with respect to the Lebesgue measure), but not continuously cohomologous. The cocycles can be made arbitrarily close to the identity and, in particular, fiber bunched. We establish continuity of a measurable conjugacy for fiber bunched cocycles under the assumption that one of them is uniformly quasiconformal. The example above shows that this assumption is close to optimal.

Definition 4.1. *A cocycle $\mathcal{B}$ is called* uniformly quasiconformal *if there exists a number $K(\mathcal{B})$ such that the quasiconformal distortion satisfies*

$$(4.1) \qquad K_{\mathcal{B}}(x, n) \overset{def}{=} \|\mathcal{B}_x^n\| \cdot \|(\mathcal{B}_x^n)^{-1}\| \le K(\mathcal{B}) \quad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

If $K_{\mathcal{B}}(x, n) = 1$ for all x and n, the cocycle is said to be conformal.

Clearly, Hölder continuous conformal cocycles are fiber bunched, and so are all sufficiently high iterates of uniformly quasiconformal cocycles.

Theorem 4.2. *Let $\mathcal{A}$ be a cocycle with a standard holonomy and let $\mathcal{B}$ be a uniformly quasiconformal Hölder cocycle. Let m be the invariant volume for f, and let C be a m-measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$. Then C coincides on a set of full measure with a continuous conjugacy that intertwines the standard holonomies of $\mathcal{A}$ and $\mathcal{B}$.*

For simplicity, when we speak of a *holonomy* for a cocycle $\mathcal{A}$ we mean a pair of a stable holonomy and an unstable holonomy, $\mathcal{H}^{\mathcal{A}} = \{\mathcal{H}^{\mathcal{A},s}, \mathcal{H}^{\mathcal{A},u}\}$. When we say that a conjugacy *intertwines* $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$ we mean mean that it intertwines both the stable and the unstable holonomies as in the following definition.

Definition 4.3. *Suppose that $H^{\mathcal{A},s}$ and $H^{\mathcal{B},s}$ are stable holonomies for cocycles $\mathcal{A}$ and $\mathcal{B}$. We say that a conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ intertwines $H^{\mathcal{A},s}$ and $H^{\mathcal{B},s}$ if*

$$(4.2) \qquad H_{x,y}^{\mathcal{A},s} = C(y) \circ H_{x,y}^{\mathcal{B},s} \circ C(x)^{-1} \quad \text{for all } x, y \in \mathcal{M} \text{ such that } y \in W^s(x).$$

Intertwining the standard holonomies of cocycles is an important property of a conjugacy C. It is clear from the proof that it implies continuity of C. Further, it can be uses to study higher regularity of the conjugacy, see [NT] for non-commutative cocycles over hyperbolic systems and [W] for real-valued cocycles over accessible partially hyperbolic systems. In contrast to real-valued cocycles, however, even continuous conjugacy between fiber bunched cocycles does not necessarily intertwine their standard holonomies.

Proposition 4.4. *For any $0 < \beta' < 1$, there exist a smooth cocycle $\mathcal{A}$ and a constant cocycle $\mathcal{B}$ over an Anosov automorphism of $\mathbb{T}^2$ that are fiber-bunched with $\beta = 1$ and conjugate via a β'-Hölder function C, but neither C nor any other conjugacy between $\mathcal{A}$ and $\mathcal{B}$ intertwines their standard holonomies.*

The next proposition gives a general sufficient condition for intertwining.

Proposition 4.5. *Suppose that cocycles $\mathcal{A}$ and $\mathcal{B}$ are β fiber bunched. Then any β-Hölder conjugacy C between them intertwines their standard holonomies.*

It is clear from the proof that it suffices to assume β-Hölder continuity of C along the stable/unstable leaves to obtain intertwining of the standard stable/unstable holonomies respectively. Conversely, intertwining β-Hölder holonomies implies β-Hölder continuity of C along the stable and unstable leaves. Then global Hölder continuity of C follows for hyperbolic f. For a partially hyperbolic f, accessibility is not known to imply global Hölder continuity of C, but a stronger assumption suffices. The diffeomorphism f is called *locally α-Hölder accessible* if there exists a number $L = L(f)$ such that for all sufficiently close $x, y \in \mathcal{M}$ there is an *su*-path

$$P = \{x = x_0,\, x_1, \ldots,\, x_L = y\} \quad \text{such that} \quad \text{dist}_{W^i}(x_{i-1}, x_i) \leq C \operatorname{dist}(x, y)^\alpha$$

for $i = 1, \ldots, L$. Here the distance between x_{i-1} and x_i is measured along the corresponding stable or unstable leaf W^i. Such accessibility implies $\alpha\beta$-Hölder continuity of C, see [KS, Corollary 3.7]. The usual accessibility implies that an *su*-path can be chosen with L and the distances $\text{dist}_{W^i}(x_{i-1}, x_i)$ uniformly bounded. If, in addition, the points x_i can be chosen to depend Hölder continuously on x and y, then Theorem 3.5 can be used to obtain global Hölder continuity of C.

Now we consider the problem of finding sufficient conditions for existence of a continuous conjugacy between two cocycles. Suppose that $H^{\mathcal{A},s}$ and $H^{\mathcal{A},u}$ are stable and unstable holonomies for a cocycle $\mathcal{A}$. Let $P = \{x_0, x_1, \ldots, x_{k-1}, x_k\}$ be an *su*-path in $\mathcal{M}$. We define the *weight* of P as

$$\mathcal{H}^{\mathcal{A},P}_{x_0,x_k} = H_{x_0,x_{k-1}} \circ \cdots \circ H_{x_1,x_2} \circ H_{x_0,x_1},$$

where $H_{x_i,x_{i+1}} = H^{s/u}_{x_i,x_{i+1}}$ if $x_{i+1} \in W^{s/u}(x_i)$. An *su-cycle* is an *su*-path in $\mathcal{M}$ with $x_0 = x_k$, and we refer to the corresponding $\mathcal{H}^{\mathcal{A},P}_{x_0}$ as the *cycle weight*. In case of real-valued cocycles, $\mathcal{H}^{\mathcal{A},P}_{x_0}$ is also referred to as the cycle functional.

The following properties are easy to verify.

Proposition 4.6. *Let $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$ be holonomies for cocycles $\mathcal{A}$ and $\mathcal{B}$ and let C be a continuous conjugacy between $\mathcal{A}$ and $\mathcal{B}$ which intertwines these holonomies. Then*

(i) *C conjugates the cycle weights of these holonomies, i.e.*

$$\mathcal{H}^{\mathcal{A},P}_x = C(x) \circ \mathcal{H}^{\mathcal{B},P}_x \circ C(x)^{-1} \quad \text{for every su-cycle } P = P_x.$$

(ii) *More generally, for any $x, y \in \mathcal{M}$ and any su-path $P_{x,y}$ from x to y,*

$$\mathcal{H}^{\mathcal{A},P}_{x,y} = C(y) \circ \mathcal{H}^{\mathcal{B},P}_{x,y} \circ C(x)^{-1} \quad \text{and hence} \quad C(y) = \mathcal{H}^{\mathcal{A},P}_{x,y} \circ C(x) \circ (\mathcal{H}^{\mathcal{B},P}_{x,y})^{-1}.$$

(iii) *C is uniquely determined by its value at any point.*

The next theorem gives a sufficient condition for existence of a continuous conjugacy intertwining holonomies. By the previous proposition, this condition is also necessary.

Theorem 4.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be cocycles with holonomies $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$. Suppose that there exist $x_0 \in \mathcal{M}$ and $C_{x_0} \in GL(V)$ such that*

(i) $\mathcal{H}_{x_0}^{\mathcal{A},P} = C_{x_0} \circ \mathcal{H}_{x_0}^{\mathcal{B},P} \circ C_{x_0}^{-1}$ *for every su-cycle P_{x_0}, and*

(ii) $\mathcal{A}_{x_0} = C_{fx_0} \circ \mathcal{B}_{x_0} \circ C_{x_0}^{-1}$, *where* $C_{fx_0} = \mathcal{H}_{x_0,fx_0}^{\mathcal{A},P} \circ C_{x_0} \circ (\mathcal{H}_{x_0,fx_0}^{\mathcal{B},P})^{-1}$ *for some su-path P_{x_0,fx_0} from x_0 to fx_0.*

Then there exists a continuous conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ with $C(x_0) = C_{x_0}$ that intertwines $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$.

We note that due to the first assumption, C_{fx_0} in (ii) does not depend on the choice of a path P_{x_0,fx_0}. If x_0 is a fixed point for f then, considering the trivial path from x_0 to $fx_0 = x_0$, we see that condition (ii) becomes $\mathcal{A}_{x_0} = C_{x_0} \circ \mathcal{B}_{x_0} \circ C_{x_0}^{-1}$, and we obtain the following corollary. Thus, in this case (i) can be viewed as a sufficient condition for extending a conjugacy from a given value at a fixed point.

Corollary 4.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be cocycles with holonomies $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$. Suppose that there exist a fixed point x_0 and $C_{x_0} \in GL(V)$ such that $\mathcal{A}_{x_0} = C_{x_0} \circ \mathcal{B}_{x_0} \circ C_{x_0}^{-1}$ and $\mathcal{H}_{x_0}^{\mathcal{A},P} = C_{x_0} \circ \mathcal{H}_{x_0}^{\mathcal{B},P} \circ C_{x_0}^{-1}$ for every su-cycle P_{x_0}. Then there exists a continuous conjugacy C between $\mathcal{A}$ and $\mathcal{B}$ with $C(x_0) = C_{x_0}$ that intertwines $H^{\mathcal{A}}$ and $H^{\mathcal{B}}$.*

Now we apply Theorem 4.7 to the question when a cocycle $\mathcal{A}$ is cohomologous to a constant cocycle. Clearly, for a constant cocycle $\mathcal{B}$ the standard holonomy is trivial, $H_{x,y}^{\mathcal{B}} = \mathrm{Id}$. Thus $\mathcal{H}^{\mathcal{B},P} = \mathrm{Id}$ for every su-cycle P and hence (i) becomes $\mathcal{H}_{x_0}^{\mathcal{A},P} = \mathrm{Id}$. Condition (ii) can be rewritten as $\mathcal{B}_{x_0} = C_{x_0}^{-1} \circ (\mathcal{H}_{x_0,fx_0}^{\mathcal{A},P})^{-1} \circ \mathcal{A}_{x_0} \circ C_{x_0}$ and so it defines a constant cocycle $\mathcal{B}$ uniquely for any choice of C_{x_0}. Thus we obtain the first part of the following corollary. It was established in [KN] for systems with local accessibility and for the standard holonomy of a cocycle satisfying a certain bunching assumption.

Corollary 4.9. *If a cocycle $\mathcal{A}$ has a holonomy $H^{\mathcal{A}}$ satisfying*

(4.3) $\qquad \mathcal{H}_{x_0}^{\mathcal{A},P} = \mathrm{Id}$ *for very su-cycle P_{x_0} based at some point $x_0 \in \mathcal{M}$,*

then there exists a continuous conjugacy between $\mathcal{A}$ and a constant cocycle $\mathcal{B}$ that intertwines $H^{\mathcal{A}}$ and the standard holonomy $H^{\mathcal{B}} = \mathrm{Id}$ for $\mathcal{B}$. Existence of such a holonomy $H^{\mathcal{A}}$ is a necessary condition for $\mathcal{A}$ to be cohomologous to a constant cocycle.

The second part of the corollary follows from Proposition 4.6 and the following observation: for any holonomy $\mathcal{H}_{x,y}^{\mathcal{B}}$ and any continuous conjugacy C between $\mathcal{A}$ and $\mathcal{B}$, the formula $C(y) \circ \mathcal{H}_{x,y}^{\mathcal{B}} \circ C(x)^{-1}$ defines a holonomy for $\mathcal{A}$. We note, however, that having the standard holonomy satisfy (4.3) is not a necessary condition for existence of a continuous conjugacy to a constant cocycle. Indeed, the cocycle $\mathcal{A}$ in Proposition 4.4 is cohomologous to the constant cocycle $\mathcal{B}$ via a continuous C, but no conjugacy intertwines their standard holonomies. This together with Corollary 4.9 implies that (4.3) does not hold for the standard holonomy of $\mathcal{A}$. Also, the standard holonomy of $\mathcal{A}$ is mapped by C to a non-standard holonomy for $\mathcal{B}$ for which (4.3) does not hold. In particular, holonomes for $\mathcal{A}$ and $\mathcal{B}$ are non-unique.

5. PROOFS

5.1. **Proof of Theorem 3.5.** Since the cocycle $\mathcal{A}$ is fiber bunched and since by (2.1) $0 < \nu(x) < \gamma(x)$, we can fix $\theta < 1$ sufficiently close to 1 so that for all $x \in \mathcal{M}$,

$$(5.1) \quad \|\mathcal{A}_x\| \cdot \|\mathcal{A}_x^{-1}\| \cdot \nu(x)^\beta \leq \theta, \quad \|\mathcal{A}_x\| \cdot \|\mathcal{A}_x^{-1}\| \cdot \hat{\nu}(x)^\beta \leq \theta, \quad \text{and} \quad (\nu(x)/\gamma(x))^\beta < \theta.$$

Since $\hat{\mu} < 1$ from (2.2) we can choose α, $0 < \alpha \leq \beta$, sufficiently close to 0 so that

$$(5.2) \qquad\qquad \theta < (\hat{\mu}(x)\nu(x))^\alpha \quad \text{for all } x \in \mathcal{M}.$$

By iterating points x, y, x'', y'' forward and using invariance of the holonomies (H3), we can assume without loss of generality that $y \in W_{\delta_0}^s(x)$, $y'' \in W_{\delta_0}^s(x'')$ for some sufficiently small $\delta_0 > 0$. We denote $E^{cu} = E^c \oplus E^u$ and let Σ_x be the exponential of the ball of radius $C_1\delta$ centered at x in $E^{cu}(x)$. Since E^{cu} is transversal to E^s, we can fix $C_1 > 0$ such that if δ is sufficiently small then for any $x \in \mathcal{M}$, Σ_x is a submanifold transversal to W^s and for any x'' with $\text{dist}(x, x'') < \delta$ there is a unique intersection point $x' = \Sigma_x \cap W^s(x'')$. If δ is sufficiently small then the distances $\text{dist}(x, x')$ and $\text{dist}(x', x'')$ are at most $C_2\text{dist}(x, x'')$, for some constant $C_2 > 0$ independent of points x, y, x'', y'', and also for each $z \in \Sigma_x$ the tangent space $T_z\Sigma_x$ is close to $E^{cu}(z)$. Similarly, we define Σ_y and y'. By taking $\delta < \delta_0$ sufficiently small we can also ensure that $x', y, y' \in B_{2\delta_0}(x)$ and $y' \in W_{2\delta_0}^s(x')$.

First we iterate the points x, x', y, y' and estimate the distances between their trajectories in the next lemma. The setting and arguments here are similar to ones in a direct proof of Hölder continuity of stable holonomies for a partially hyperbolic system, cf. [W, Proposition 5.2]. We denote $x_k = f^k x$, and

$$(5.3) \qquad \nu_k(x) = \nu(f^{k-1}x) \cdots \nu(fx)\,\nu(x) = \nu(x_{k-1}) \cdots \nu(x_1)\,\nu(x_0).$$

We will use similar notations for x', y, and y' as well as for the functions $\hat{\nu}$, $\hat{\mu}$, and γ. We choose n so that $\text{dist}(x, x') \approx \nu_n(x)\,\hat{\mu}_n(x)$. More precisely, we take n to be the largest integer satisfying the first inequality in

$$(5.4) \qquad\qquad \text{dist}(x, x') \leq \nu_n(x)\,\hat{\mu}_n(x) \leq C'\text{dist}(x, x')$$

This implies the second inequality with some constant C' independent of x, x'.

Lemma 5.1. *Let n be chosen according to (5.4). Then there exists M such that*

    (a) *$\text{dist}_{W^s}(x_n, y_n) \leq M\nu_n(x)$ and $\text{dist}_{W^s}(x'_n, y'_n) \leq M\nu_n(x)$;*

    (b) *$\text{dist}(x_k, x'_k) \leq \nu_n(x)\,\hat{\mu}_{n-k}(x_k)$ and $\text{dist}(y_k, y'_k) \leq M\nu_n(x)\gamma_{n-k}(x_k)^{-1}$*
       for $0 \leq k \leq n$.

Proof. By continuity of the functions ν, $\hat{\mu}$, and γ from (2.1) and (2.2), there exists $0 < r < 1$ such for any point $p \in \mathcal{M}$ the value at p gives the corresponding estimate for any $q \in B_r(x)$. It will be clear from the estimates that by taking δ_0 and δ small enough, which forces n to be large enough, we can ensure that $x'_k, y_k, y'_k \in B_r(x_k)$ for

each $0 \leq k \leq n$. The first part of (b) follows since $\hat{\mu}^{-1}$ bounds above the maximal expansion of f:

$$\text{dist}(x_k, x'_k) \leq \text{dist}(x, x')\, \hat{\mu}_k(x)^{-1} \leq \nu_n(x)\, \hat{\mu}_n(x)\, \hat{\mu}_k(x)^{-1} = \nu_n(x)\, \hat{\mu}_{n-k}(x_k).$$

Since $y \in W^s_{\delta_0}(x)$ and $y' \in W^s_{2\delta_0}(x')$ we obtain

$$\text{dist}_{W^s}(x_n, y_n) \leq \delta_0\, \nu_n(x) \quad \text{and} \quad \text{dist}_{W^s}(x'_n, y'_n) \leq 2\delta_0\, \nu_n(x).$$

Choosing $M = 3\delta_0 + 1$ we obtain part (a) and the estimate

$$\text{dist}(y_n, y'_n) \leq \text{dist}(x_n, x'_n) + \text{dist}(x_n, y_n) + \text{dist}(x'_n, y'_n) \leq M\, \nu_n(x).$$

Since γ is less than the strongest contraction along E^{cu}, we obtain the second part of (b):

$$\text{dist}(y_k, y'_k) \leq \text{dist}(y_n, y'_n)\gamma_{n-k}(x_k)^{-1} \leq M\, \nu_n(x)\gamma_{n-k}(x_k)^{-1}$$

for $k = 0, 1, \ldots, n$. For this we note that the transversals Σ_x and Σ_y are chosen close to E^{cu} and that their forward iterates $f^k(\Sigma_x)$ and $f^k(\Sigma_y)$ will remain close to E^{cu}. $\qquad \square$

Now we estimate the holonomies. For simplicity, in this proof we use H for the standard stable holonomy $H^{\mathcal{A},s}$. Our goal is to show that

$$(5.5) \qquad \qquad \|H_{x',y'} \circ H_{x,y}^{-1} - \text{Id}\| \leq C_{14}\text{dist}(x, x')^\alpha.$$

Note that all relevant holonomies between points x, x', x'', y, y', y'' lie in a compact subset of $GL(V)$. Thus, once (5.5) is established, Lemma 2.1 implies a Hölder estimate for $d(H_{x,y}, H_{x',y'})$ similar to (5.5). Also, since $H_{x'',y'}^{-1} \circ H_{x',y'} = H_{x',x''}$, (H4) and the estimate $\text{dist}(x', x'') \leq C_2\text{dist}(x, x'')$ give a β-Hölder estimate for $d(H_{x',y'}, H_{x'',y'})$. Similarly, $\text{dist}(y', y'') \leq C_2\text{dist}(y, y'')$ gives a β-Hölder estimate for $d(H_{x'',y'}, H_{x'',y''})$. We conclude that (5.5) yields the desired α-Hölder estimate for $d(H_{x,y}, H_{x'',y''})$ and proves the theorem. To prove (5.5) we write

$$H_{x',y'} \circ H_{x,y}^{-1} = \left((\mathcal{A}_{y'}^n)^{-1} \circ H_{x'_n,y'_n} \circ \mathcal{A}_{x'}^n\right) \circ \left((\mathcal{A}_y^n)^{-1} \circ H_{x_n,y_n} \circ \mathcal{A}_x^n\right)^{-1} =$$

$$(5.6) \qquad = (\mathcal{A}_{y'}^n)^{-1} \circ H_{x'_n,y'_n} \circ \left(\mathcal{A}_{x'}^n \circ (\mathcal{A}_x^n)^{-1}\right) \circ (H_{x_n,y_n})^{-1} \circ \mathcal{A}_y^n =$$

$$= (\mathcal{A}_{y'}^n)^{-1} \circ (\text{Id} + \Delta_1) \circ (\text{Id} + \Delta_2) \circ (\text{Id} + \Delta_3) \circ \mathcal{A}_y^n,$$

where

$$\Delta_1 = H_{x'_n,y'_n} - \text{Id}, \qquad \Delta_2 = \mathcal{A}_{x'}^n \circ (\mathcal{A}_x^n)^{-1} - \text{Id}, \qquad \Delta_3 = H_{y_n,x_n} - \text{Id}.$$

By (H4) and Lemma 5.1(a) we have

$$\|\Delta_1\| = \|H_{x'_n,y'_n} - \text{Id}\| \leq K\text{dist}_{W^s}(x'_n, y'_n)^\beta \leq KM^\beta\nu_n(x)^\beta,$$

and similarly $\|\Delta_3\| \leq KM^\beta\nu_n(x)^\beta$. Also, by Lemma 5.2 below we have

$$\|\Delta_2\| = \|\mathcal{A}_{x'}^n \circ (\mathcal{A}_x^n)^{-1} - \text{Id}\| \leq C_7\nu_n(x)^\beta.$$

Therefore, from (5.6) we obtain

$$(5.7) \quad \|H_{x',y'} \circ H_{x,y}^{-1} - \mathrm{Id}\| \leq \|(\mathcal{A}_{y'}^n)^{-1} \circ \mathcal{A}_y^n - \mathrm{Id}\| + \|(\mathcal{A}_{y'}^n)^{-1}\| \cdot \|\mathcal{A}_y^n\| \cdot C_{12}\nu_n(x)^{\beta}.$$

Equation (5.7) and Lemma 5.4 imply that

$$\|H_{x',y'} \circ H_{x,y}^{-1} - \mathrm{Id}\| \leq C_{11}\theta^n + C_9\,\theta^n \nu_n(x)^{-\beta}C_{12}\,\nu_n(x)^{\beta} \leq C_{13}\theta^n,$$

and by the choices of α and n, (5.2) and (5.4), we conclude that

$$\|H_{x',y'} \circ H_{x,y}^{-1} - \mathrm{Id}\| \leq C_{13}\theta^n \leq C_{13}\,(\hat{\mu}_n(x)\nu_n(x))^{\alpha} \leq C_{13}\,(C'\mathrm{dist}(x,x'))^{\alpha}.$$

This completes the proof of the theorem modulo Lemmas 5.1, 5.2, and 5.4.

Lemma 5.2. $\|\mathcal{A}_{x'}^n \circ (\mathcal{A}_x^n)^{-1} - \mathrm{Id}\| \leq C_7\nu_n(x)^{\beta}.$

Proof. We rewrite $\mathcal{A}_{x'}^n \circ (\mathcal{A}_x^n)^{-1}$ as follows

$$\mathcal{A}_{x'}^n \circ (\mathcal{A}_x^n)^{-1} = \mathcal{A}_{x_1'}^{n-1} \circ \mathcal{A}_{x_0'} \circ (\mathcal{A}_{x_0})^{-1} \circ (\mathcal{A}_{x_1}^{n-1})^{-1} =$$

$$= \mathcal{A}_{x_1'}^{n-1} \circ (\mathrm{Id} + r_0) \circ (\mathcal{A}_{x_1}^{n-1})^{-1} =$$

$$(5.8) \qquad = \mathcal{A}_{x_1'}^{n-1} \circ (\mathcal{A}_{x_1}^{n-1})^{-1} + \mathcal{A}_{x_1'}^{n-1} \circ r_0 \circ (\mathcal{A}_{x_1}^{n-1})^{-1} = \cdots =$$

$$= \mathrm{Id} + \sum_{i=1}^{n} \mathcal{A}_{x_i'}^{n-i} \circ r_{i-1} \circ (\mathcal{A}_{x_i}^{n-i})^{-1}, \quad \text{where } r_i = \mathrm{Id} - (\mathcal{A}_{x_i'})^{-1} \circ \mathcal{A}_{x_i}.$$

First we estimate $\|r_i\|$ using boundedness of $\|(\mathcal{A}_{x_i'})^{-1}\|$ and Lemma 5.1 (b):

$$(5.9) \qquad \begin{aligned} \|r_i\| = \|\mathrm{Id} - (\mathcal{A}_{x_i'})^{-1} \circ \mathcal{A}_{x_i}\| &\leq \|(\mathcal{A}_{x_i'})^{-1}\| \cdot \|\mathcal{A}_{x_i'} - \mathcal{A}_{x_i}\| \\ &\leq C_3 \cdot \mathrm{dist}(x_i, x_i')^{\beta} \leq C_3\,(\nu_n(x)\,\hat{\mu}_{n-i}(x_i))^{\beta}. \end{aligned}$$

Next we estimate $\|\mathcal{A}_{x_i'}^{n-i}\| \cdot \|(\mathcal{A}_{x_i}^{n-i})^{-1}\|$. Using Hölder continuity of $\mathcal{A}$ we obtain

$$\frac{\|\mathcal{A}_{x_k'}\|}{\|\mathcal{A}_{x_k}\|} = \frac{\|\mathcal{A}_{x_k} + \mathcal{A}_{x_k'} - \mathcal{A}_{x_k}\|}{\|\mathcal{A}_{x_k}\|} \leq 1 + \frac{\|\mathcal{A}_{x_k'} - \mathcal{A}_{x_k}\|}{\|\mathcal{A}_{x_k}\|} \leq 1 + C_4\,\mathrm{dist}(x_k, x_k')^{\beta}.$$

Hence we obtain using (5.1) that

$$\|\mathcal{A}_{x_i'}^{n-i}\| \|(\mathcal{A}_{x_i}^{n-i})^{-1}\| \leq \prod_{k=i}^{n-1} \|\mathcal{A}_{x_k'}\| \cdot \prod_{k=i}^{n-1} \|(\mathcal{A}_{x_k})^{-1}\| \leq \prod_{k=i}^{n-1} \|\mathcal{A}_{x_k}\| \|(\mathcal{A}_{x_k})^{-1}\| \cdot \prod_{k=i}^{n-1} \frac{\|\mathcal{A}_{x_k'}\|}{\|\mathcal{A}_{x_k}\|}$$

$$\leq \prod_{k=i}^{n-1} \theta\,\hat{\nu}(x_k)^{-\beta} \cdot \prod_{k=i}^{n-1} \left(1 + C_4\,(\mathrm{dist}(x_k, x_k'))^{\beta}\right) \leq$$

$$\leq \theta^{n-i}\hat{\nu}_{n-i}(x_i)^{-\beta} \cdot \prod_{k=i}^{n-1} \left(1 + C_4\,(\nu_n(x)\hat{\mu}_{n-k}(x_k))^{\beta}\right) \leq C_5\,\theta^{n-i}\hat{\nu}_{n-i}(x_i)^{-\beta},$$

as the product is uniformly bounded in n and i since $\nu, \hat{\mu} < 1$. In particular,

$$(5.10) \qquad \|\mathcal{A}_{x'}^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| \leq C_5\,\theta^n\,\hat{\nu}_n(x)^{-\beta}.$$

Now using (5.8) and (5.9) we conclude that

$$\|\mathcal{A}_{x'}^n \circ (\mathcal{A}_x^n)^{-1} - \mathrm{Id}\| \leq \sum_{i=1}^n \|\mathcal{A}_{x_i'}^{n-i} \circ r_{i-1} \circ (\mathcal{A}_{x_i}^{n-i})^{-1}\| \leq$$

$$\leq \sum_{i=1}^n C_3 \, \nu_n(x)^\beta \hat{\mu}_{n-i+1}(x_{i-1})^\beta \cdot C_5 \, \theta^{n-i} \hat{\nu}_{n-i}(x_i)^{-\beta} \leq$$

$$\leq C_6 \, \nu_n(x)^\beta \sum_{i=1}^n \theta^{n-i} \, (\hat{\nu}_{n-i}(x_i)^{-1} \hat{\mu}_{n-i}(x_i))^\beta \leq C_7 \, \nu_n(x)^\beta$$

since $\theta < 1$ and $\hat{\mu} < \hat{\nu}$. This completes the proof of Lemma 5.2. $\qquad\square$

Lemma 5.3. *There exists C_8 such that if $w \in W_r^s(z)$ then for any $k \in \mathbb{N}$*

$$\| \mathcal{A}_w^k \| \leq C_8 \| \mathcal{A}_z^k \| \qquad and \qquad \|(\mathcal{A}_w^k)^{-1}\| \leq C_8 \|(\mathcal{A}_z^k)^{-1}\|.$$

Proof. From (H3) we have that $\mathcal{A}_w^k = H_{f^k z, f^k w} \circ \mathcal{A}_z^k \circ H_{z,w}^{-1}$ and obtain the first inequality since the norms of $H_{z,w}$ and $H_{z,w}^{-1}$ are bounded uniformly in $z \in \mathcal{M}$ and $w \in W_r^s(z)$ by compactness. The second one is established similarly. $\qquad\square$

Lemma 5.4. $\|\mathcal{A}_{x'}^n\| \cdot \|(\mathcal{A}_x^n)^{-1}\| \leq C_9 \theta^n \nu_n(x)^{-\beta}$ *and* $\|(\mathcal{A}_{y'}^n)^{-1} \circ \mathcal{A}_y^n - Id\| \leq C_{11}\theta^n$.

Proof. First we claim that $\|\mathcal{A}_{x'}^i\| \cdot \|(\mathcal{A}_x^i)^{-1}\| \leq C_5 \theta^i \nu_i(x)^{-\beta}$ for $0 \leq i \leq n$. This is obtained in the same way as (5.10) using the first inequality in (5.1) instead of the second one. Applying the previous lemma we also obtain

$$\|(\mathcal{A}_{y'}^i)^{-1}\| \leq C_8 \|(\mathcal{A}_{x'}^k)^{-1}\| \qquad and \qquad \|\mathcal{A}_y^i\| \leq C_8 \|\mathcal{A}_x^i\|$$

for all $i \in \mathbb{N}$. We conclude that for each $0 \leq i \leq n$,

$$\|(\mathcal{A}_{y'}^i)^{-1}\| \cdot \|\mathcal{A}_y^i\| \leq C_9 \, \theta^i \, \nu_i(x)^{-\beta}$$

giving, in particular, the first inequality in the lemma.

Similarly to (5.8) and (5.9) we obtain using Lemma 5.1 (b) that

$$(\mathcal{A}_{y'}^n)^{-1} \circ \mathcal{A}_y^n = \mathrm{Id} + \sum_{i=0}^{n-1}(\mathcal{A}_{y'}^i)^{-1} \circ r_i \circ \mathcal{A}_y^i, \quad \text{where } r_i = \mathrm{Id} - (\mathcal{A}_{y_i'})^{-1} \circ \mathcal{A}_{y_i}$$

satisfy $\quad \|r_i\| = \|(\mathcal{A}_{y_i'})^{-1} \circ \mathcal{A}_{y_i} - \mathrm{Id}\| \leq C_3 \, \mathrm{dist}(y_i, y_i')^\beta \leq MC_3\nu_n(x)^\beta \gamma_{n-i}(x_i)^{-\beta}.$

Using that $\nu(x)^\beta (\gamma(x)^\beta \theta)^{-1} < 1$ by (5.1), we conclude that

$$\|(\mathcal{A}_{y'}^n)^{-1} \circ \mathcal{A}_y^n - \mathrm{Id}\| \leq \sum_{i=0}^{n-1} \|(\mathcal{A}_{y'}^i)^{-1}\| \cdot \|\mathcal{A}_y^i\| \cdot \|r_i\| \leq$$

$$\leq MC_3C_9 \sum_{i=0}^{n-1} \theta^i \nu_i(x)^{-\beta}\nu_n(x)^\beta \gamma_{n-i}(x_i)^{-\beta} \leq C_{10}\theta^n \sum_{i=0}^{n-1} \theta^{i-n}\nu_{n-i}(x_i)^\beta \gamma_{n-i}(x_i)^{-\beta} \leq C_{11}\theta^n.$$

$\qquad\square$

5.2. **Proof of Proposition 3.6.** Let $\rho_1 < \cdots < \rho_l$ be the distinct moduli of the eigenvalues of the matrix A. Let $\mathbb{R}^d = E_1 \oplus \cdots \oplus E_l$ be the corresponding splitting into direct sums of the generalized eigenspaces, and let $\mathcal{A}_i = \mathcal{A}|_{E_i}$. Then for any $\epsilon > 0$ there exists K_ϵ such that

$$K_\epsilon^{-1}(\rho_i - \epsilon)^n \le \|\mathcal{A}_i^n v\| \le K_\epsilon(\rho_i + \epsilon)^n \quad \text{for any unit vector } v \in E_i.$$

Then any sufficiently C^0 small Hölder continuous perturbation $\mathcal{B}$ of $\mathcal{A}$ has a Hölder continuous invariant splitting with similar estimates for the corresponding restrictions $\mathcal{B}_i$. It follows that $\mathcal{B}_i$'s are close to conformal and satisfy the weaker fiber bunching condition (3.2). Hence by Remark 3.3 $\mathcal{B}_i$'s have standard holonomies, which combine into the standard holonomy for $\mathcal{B}$. We note, however, that the Hölder exponent of the splitting and of the resulting holonomy may be lower than that of $\mathcal{B}$.

5.3. **Proof of Theorem 4.2.** Let $H^\mathcal{A}$ be the standard holonomies for $\mathcal{A}$, which exist by the assumption. Since $\mathcal{B}$ is uniformly quasiconformal, it satisfies the weaker fiber bunching condition (3.2). Thus, by Remark 3.3, $\mathcal{B}$ has standard holonomies, which we denote by $H^\mathcal{B}$.

Our main goal is to show that C intertwines the holonomies of $\mathcal{A}$ and $\mathcal{B}$ on a set of full measure. More precisely, for the stable holonomies we will show that there exists a subset Y of $\mathcal{M}$ with $m(Y) = 1$ such that (4.2) holds for all $x, y \in Y$ such that $y \in W^s(x)$. A similar statement holds for the unstable holonomies.

By the assumption, there is a set of full measure $Y_1 \subset \mathcal{M}$ such that for all $x \in Y_1$, $\mathcal{A}_x = C(fx) \circ \mathcal{B}_x \circ C(x)^{-1}$. Since the function C is m-measurable, by Lusin's theorem there exists a compact set $S \subset \mathcal{M}$ with $m(S) > 1/2$ such that C is uniformly continuous on S. It follows that $\|C\|$ and $\|C^{-1}\|$ are bounded on S. Let Y_2 be the set of points in $\mathcal{M}$ for which the frequency of visiting S equals $m(S) > 1/2$. By Birkhoff ergodic theorem, $m(Y_2) = 1$.

Let $Y = Y_1 \cap Y_2$. Clearly, $m(Y) = 1$ and we can assume that the sets Y_1, Y_2, Y are f-invariant. Suppose that $x, y \in Y$ and $y \in W_R^s(x)$ for some fixed radius R. Then

(5.11)
$$\begin{aligned}
(\mathcal{A}_y^n)^{-1} \circ \mathcal{A}_x^n &= \left(C(f^n y) \circ \mathcal{B}_y^n \circ C(y)^{-1}\right)^{-1} \circ C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1} = \\
&= C(y) \circ (\mathcal{B}_y^n)^{-1} \circ C(f^n y)^{-1} \circ C(f^n x) \circ \mathcal{B}_x^n \circ C(x)^{-1} = \\
&= C(y) \circ (\mathcal{B}_y^n)^{-1} \circ (\mathrm{Id} + \Delta_n) \circ \mathcal{B}_v^n \circ C(x)^{-1} = \\
&= C(y) \circ (\mathcal{B}_y^n)^{-1} \circ \mathcal{B}_x^n \circ C(x)^{-1} + C(y) \circ (\mathcal{B}_y^n)^{-1} \circ \Delta_n \circ \mathcal{B}_x^n \circ C(x)^{-1}.
\end{aligned}$$

We will show that the second term in the last line tends to 0 along a subsequence. First we estimate the norm of Δ_n.

(5.12) $$\|\Delta_n\| = \|C(f^n y)^{-1} \circ C(f^n x) - \mathrm{Id}\| \le \|C(f^n y)^{-1}\| \cdot \|C(f^n x) - C(f^n y)\|.$$

Since $x, y \in Y_2 \subset Y$, there exists a sequence $\{n_i\}$ such that $f^{n_i} x, f^{n_i} y \in S$ for all i. Since $y \in W_R^s(x)$, $\mathrm{dist}(f^{n_i} x, f^{n_i} y) \to 0$ and hence $\|C(f^{n_i} x) - C(f^{n_i} y)\| \to 0$ by

uniform continuity of C on S. As $\|C^{-1}\|$ is uniformly bounded on Y, (5.12) implies

$$\|\Delta_{n_i}\| \to 0 \quad \text{as } i \to \infty$$

Using Lemma 5.3 and quasiconformality of $\mathcal{B}$ we also obtain that

$$(5.13) \qquad \|(\mathcal{B}_y^n)^{-1}\| \cdot \|\mathcal{B}_x^n\| \le \|(\mathcal{B}_y^n)^{-1}\| \cdot C_8 \|\mathcal{B}_y^n\| \le C_8 \, K(\mathcal{B})$$

for all $x \in \mathcal{M}$ and $y \in W_R^s(x)$. Now it follows that

$$\|C(y) \circ (\mathcal{B}_y^{n_i})^{-1} \circ \Delta_{n_i} \circ \mathcal{B}_x^{n_i} \circ C(x)^{-1}\| \to 0 \quad \text{as } i \to \infty.$$

Since the holonomies $H^{A,s}$ and $H^{B,s}$ are standard, i.e. satisfy (3.1), passing to the limit in (5.11) along the sequence n_i yields

$$(5.14) \qquad H_{x,y}^{A,s} = C(y) \circ H_{x,y}^{B,s} \circ C(x)^{-1} \quad \text{for all } x, y \in Y \text{ such that } y \in W_R^s(x).$$

We conclude that C intertwines the holonomies H^A and H^B on a set of full measure.

It follows that $C(y) = H_{x,y}^{A,s} \circ C(x) \circ (H_{x,y}^{B,s})^{-1}$ and, by continuity of holonomies, we conclude that C is so called *essentially s-continuous* in the sense of [ASV]. Similarly, C is *essentially u-continuous*. By the assumption on the base system (f is center bunched and accessible), [ASV, Theorem E] implies that C coincides on a set of full measure with a continuous function $\tilde{C}$. It follows that $\tilde{C}$ is a conjugacy between $\mathcal{A}$ and $\mathcal{B}$ and, by (5.14), intertwines H^A and H^B.

5.4. **Proof of Proposition 4.5.** As in the proof of Theorem 4.2 we obtain (5.11). Since C is β-Hölder, for any $x \in \mathcal{M}$ and $y \in W_R^s(x)$

$$\|\Delta_n\| \le \|C(f^n y)^{-1}\| \cdot \|C(f^n x) - C(f^n y)\|$$
$$\le K_1 \text{dist}(f^n x, f^n y)^\beta \le K_2 \, \nu_n(x)^\beta \text{dist}(x, y)^\beta.$$

Using fiber bunching of $\mathcal{B}$ we choose $\theta < 1$ as in (5.1), and by Lemma 5.3 we obtain

$$\|(\mathcal{B}_y^n)^{-1} \circ \Delta_n \circ \mathcal{B}_x^n\| \le \|(\mathcal{B}_y^n)^{-1}\| \cdot \|\mathcal{B}_x^n\| \cdot \|\Delta_n\|$$
$$\le C_8 \|(\mathcal{B}_x^n)^{-1}\| \cdot \|\mathcal{B}_x^n\| \cdot K_2 \, \nu_n(x)^\beta \text{dist}(x, y)^\beta \le K_3 \, \theta^n \, \text{dist}(x, y)^\beta.$$

It follows that the second term in last line of (5.11) tends to 0 as $n \to \infty$ for every $x \in \mathcal{M}$ and $y \in W_R^s(x)$. Passing to the limit in (5.11) we conclude that C intertwines the standard holonomies of $\mathcal{A}$ and $\mathcal{B}$.

5.5. **Proof of Proposition 4.4.** We use the construction described in [KN, Theorem 5.5.3] which was based on an example by R. de la Llave [dlL]. Let f be an Anosov automorphism of $\mathbb{T}^2$ with eigenvalues $\lambda > 1$ and λ^{-1}. We fix a number r, where $\beta' < r < 1$, and set $\mu = \lambda^r$. We consider smooth $GL(2, \mathbb{R})$-valued cocycles over f

$$\mathcal{B} = \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{A}(x) = \begin{bmatrix} \mu & \phi(x) \\ 0 & 1 \end{bmatrix}$$

Since $\mu < \lambda$, the cocycle $\mathcal{B}$ is fiber bunched with $\beta = 1$. We take ϕ sufficiently small, so that $\mathcal{A}$ is sufficiently C^0 close to $\mathcal{B}$ and hence it is also fiber bunched with

$\beta = 1$. Hence both $\mathcal{A}$ and $\mathcal{B}$ have standard stable and unstable holonomies which are Lipschitz along the leaves of the corresponding foliation, i.e. satisfy (H4) with $\beta = 1$.

By Theorem 5.5.3 in [KN], for any $\epsilon > 0$ there exists $\phi(x)$ such that $\mathcal{A}$ and $\mathcal{B}$ are cohomologous via a $C^{r-\epsilon}$ conjugacy C, but not by a $C^{r+\epsilon}$ conjugacy. We take ϵ such that $\beta' < r - \epsilon$ and $r + \epsilon < 1$ so that C is β' Hölder, but there is no Lipchitz conjugacy between $\mathcal{A}$ and $\mathcal{B}$. Then no conjugacy $\tilde{C}$ can intertwine the standard holonomies of $\mathcal{A}$ and $\mathcal{B}$. Indeed, otherwise $\tilde{C}$ would be Lipschitz along the stable and unstable leaves of f, since so are the standard holonomies, and hence would be Lipschitz on $\mathbb{T}^2$.

In this example, the low regularity of C is due to the low regularity of the unique invariant expanding sub-bundle $\mathcal{V}$ for $\mathcal{A}$, which has to be mapped by C to the first coordinate line. In fact, C and $\mathcal{V}$ are smooth along the stable leaves of f, and C intertwines the standard stable holonomies of $\mathcal{A}$ and $\mathcal{B}$, but not the unstable ones.

5.6. **Proof of Theorem 4.7.** In the proof we will use x in place of x_0 to simplify notations. We define $C(x) = C_x$, and then for every $y \in \mathcal{M}$ we define

$$C(y) = \mathcal{H}_{x,y}^{\mathcal{A},P} \circ C(x) \circ (\mathcal{H}_{x,y}^{\mathcal{B},P})^{-1},$$

where $P_{x,y}$ is an su-path from x to y. Note that $C \mapsto \mathcal{H}_{x,y}^{\mathcal{A},P} \circ C \circ (\mathcal{H}_{x,y}^{\mathcal{B},P})^{-1}$ defines a map from the group $GL(V)$ of operators on the fiber at x to the one on the fiber at y, and that a concatenation of paths corresponds to the composition of the maps. Therefore, it is easy to check that the assumption (i) implies that $C(y)$ is independent of the su-path P and hence is well-defined. In particular, it follows that for any $y, z \in \mathcal{M}$ and any su-path $P_{y,z}$ from y to z

$$(5.15) \qquad C(z) = \mathcal{H}_{y,z}^{\mathcal{A},P} \circ C(y) \circ (\mathcal{H}_{y,z}^{\mathcal{B},P})^{-1}.$$

Hence continuity of holonomies implies that the function C is continuous along the stable and unstable foliations of f. Since f is accessible, this implies continuity of C on $\mathcal{M}$.

It remains to show that C satisfies the cohomological equation. Consider any $y \in \mathcal{M}$ and fix an su-path $P = P_{x,y}$ from x to y. Then fP is an su-path from fx to fy. By property (H3) of holonomies we obtain using (5.15) with $z = fy$ and $y = fx$ that

$$C(fy) = \mathcal{H}_{fx,fy}^{\mathcal{A},fP} \circ C(fx) \circ (\mathcal{H}_{fx,fy}^{\mathcal{B},fP})^{-1} =$$
$$= \mathcal{A}_y \circ \mathcal{H}_{x,y}^{\mathcal{A},P} \circ \mathcal{A}_x^{-1} \circ C(fx) \circ \mathcal{B}_x \circ (\mathcal{H}_{x,y}^{\mathcal{B},P})^{-1} \circ \mathcal{B}_y^{-1}.$$

By assumption (ii) and (5.15),

$$C(fy) = \mathcal{A}_y \circ \mathcal{H}_{x,y}^{\mathcal{A},P} \circ C(x) \circ (\mathcal{H}_{x,y}^{\mathcal{B},P})^{-1} \circ \mathcal{B}_y^{-1} = \mathcal{A}_y \circ C(y) \circ \mathcal{B}_y^{-1},$$

and we conclude that C is a conjugacy.

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