LINEAR COCYCLES OVER HYPERBOLIC SYSTEMS AND CRITERIA OF CONFORMALITY

BORIS KALININ AND VICTORIA SADOVSKAYA
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ABSTRACT. In this paper, we study Hölder-continuous linear cocycles over transitive Anosov diffeomorphisms. Under various conditions of relative pinching we establish properties including existence and continuity of measurable invariant subbundles and conformal structures. We use these results to obtain criteria for cocycles to be isometric or conformal in terms of their periodic data. We show that if the return maps at the periodic points are, in a sense, conformal or isometric then so is the cocycle itself with respect to a Hölder-continuous Riemannian metric.

1. INTRODUCTION

Linear cocycles over a dynamical system $f: \mathcal{M} \to \mathcal{M}$ appear naturally in various areas of dynamics and applications. Examples include derivative cocycles as well as stochastic processes and random matrices. A linear cocycle over f is an automorphism F of a vector bundle $\mathcal{E}$ over $\mathcal{M}$ that projects to f. In the case of a trivial vector bundle $\mathcal{M} \times \mathbb{R}^d$, any linear cocycle can be identified with a matrix-valued function $A: \mathcal{M} \to GL(d, \mathbb{R})$ via $F(x, v) = (f(x), A(x)v)$.

In this paper we take f to be a transitive Anosov diffeomorphism of a compact manifold $\mathcal{M}$. However, our techniques can be applied to hyperbolic sets and some symbolic dynamical systems. We consider a finite-dimensional vector bundle $P: \mathcal{E} \to \mathcal{M}$ and a Hölder-continuous linear cocycle $F: \mathcal{E} \to \mathcal{E}$ over f (see Section 3 for definitions). One of the primary examples of such cocycles comes from the differential Df or its restriction to a Hölder-continuous invariant subbundle of $T\mathcal{M}$. Such cocycles play a crucial role in smooth dynamics of hyperbolic systems.

We establish several properties of Hölder-continuous linear cocycles under various conditions of relative pinching. These properties, which include existence and continuity of measurable invariant subbundles and conformal structures, are of independent interest and we formulate them in the next section. As the main applications we obtain conditions on F at the periodic points of f

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which guarantee that the cocycle is conformal or isometric. Our first theorem establishes a general criterion, and Theorem 1.3 below gives a stronger result specific to bundles with 2-dimensional fibers. We note that the assumptions of the theorems are independent of the choice of a continuous Riemannian metric on $\mathscr{E}$.

THEOREM 1.1. *Let $F\colon \mathscr{E} \to \mathscr{E}$ be a Hölder-continuous linear cocycle over a transitive C^2 Anosov diffeomorphism f. Suppose that there exists a constant C_{per} such that for each periodic point p, the quasiconformal distortion satisfies*

$$K_F(p,n) := \|F_p^n\| \cdot \left\|(F_p^n)^{-1}\right\| \le C_{per}$$

whenever $f^n p = p$. Then, F is conformal with respect to a Hölder-continuous Riemannian metric on $\mathscr{E}$. Moreover, if there exists a constant C'_{per} such that for each periodic point p

$$\max\{\|F_p^n\|, \left\|(F_p^n)^{-1}\right\|\} \le C'_{per}$$

whenever $f^n p = p$, then F is an isometry with respect to a Hölder-continuous Riemannian metric on $\mathscr{E}$.

For a cocycle on a trivial bundle $\mathscr{M} \times \mathbb{R}^d$ given by $A\colon \mathscr{M} \to GL(d,\mathbb{R})$, the theorem implies cohomology to a cocycle with values in the conformal or orthogonal subgroup. This means that there exists a Hölder-continuous function $C\colon \mathscr{M} \to GL(d,\mathbb{R})$ such that $B(x) = C^{-1}(f(x))A(x)C(x)$ is in the corresponding subgroup for all $x \in \mathscr{M}$. The matrix $C(x)$ can be obtained as the unique positive square root of the symmetric positive-definite matrix that defines the Riemannian metric at x.

Continuous reduction to orthogonal or conformal cocycles is useful, in particular, since cocycles with values in compact groups are relatively well understood. Some definitive results on cohomology of such cocycles were obtained in [15, 17, 18, 21]. These results can be easily extended to cocycles with values in the conformal group. However, the question of existence of such a reduction is highly nontrivial. Even under the much stronger assumption that $\|F_x^n\|$ are uniformly bounded for all $x \in \mathscr{M}$ and $n \in \mathbb{Z}$, the question remained open since it was formulated in [21]. Under assumptions on periodic data only, no reduction was known until recent progress in [6] even for the simplest case when $F_p^n = \mathrm{Id}$ for all periodic points.

Theorem 1.1 can be compared to recent results by de la Llave and Windsor [14, Theorems 6.3, 6.8] who obtained similar conclusions for the cocycle given by the restriction of the derivative of an Anosov map to a Hölder-continuous invariant subbundle of $T\mathscr{M}$. The main difference is that our theorem does not have any extra assumptions on growth or pinching of the cocycle which are present in [14] and in most other results in the theory of noncommutative cocycles.

Conformality arises naturally in connection with smooth rigidity for Anosov systems [22, 9, 24, 11, 20, 7], in particular, some of our results are motivated by the study of derivative cocycles in [8]. It is well known that a C^1 small perturbation g of an Anosov diffeomorphism f is conjugate to f by a Hölder

homeomorphism h. If h is C^1 then Df_p^n and Dg_{hp}^n are conjugate by Dh_p for any periodic point p. Therefore the conjugacy of Df_p^n and Dg_{hp}^n gives a necessary condition for h to be C^1. This condition is sufficient for systems with one-dimensional stable and unstable distributions, but not in higher dimensions [11]. The question of sufficiency of this condition is often referred to as local rigidity. Knowing that Df and Dg are conformal on the stable/unstable distribution, or on a smaller invariant distribution, helps bootstrap regularity of h along the corresponding foliation. Thus, given certain conformality of f one would like to obtain similar conformality of g. This motivates the question whether a cocycle is conformal given that the return maps F_p^n at the periodic points are conjugate to conformal maps. This question is also natural from the point of view of cohomology of cocycles. The following proposition shows, however, that the answer is negative in dimension higher than two.

PROPOSITION 1.2. *Let $f\colon \mathcal{M} \to \mathcal{M}$ be an Anosov diffeomorphism and $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, $d \geq 3$. For any $\epsilon > 0$, there exists a Lipschitz continuous linear cocycle $F\colon \mathcal{E} \to \mathcal{E}$, which is ϵ-close to the identity, such that for all periodic points $p \in \mathcal{M}$ the return maps $F_p^n\colon \mathcal{E}_p \to \mathcal{E}_p$ are conjugate to orthogonal maps, but F is not conformal with respect to any continuous Riemannian metric on $\mathcal{E}$.*

We note that, for a given p, having a uniform bound on $\|F_p^n\| \cdot \|(F_p^n)^{-1}\|$ for all periods n is equivalent to each of the following three statements: F_p^n is diagonalizable over $\mathbb{C}$ with its eigenvalues equal in modulus; F_p^n is conjugate to a conformal linear map; there exists an inner product on $\mathcal{E}_p$ with respect to which F_p^n is conformal. In fact, the periodic assumption in the first part of Theorem 1.1 is equivalent to having such inner products for all periodic points uniformly bounded. In the context of local rigidity, additional assumptions were made to ensure such boundedness, for example that all return maps F_p^n are scalar multiples of the identity [11, 7, 12]. Our next result for two-dimensional bundles does not require any extra assumptions. It can be applied, in particular, to the study of local rigidity without restrictive assumptions on the structure of Df_p^n.

THEOREM 1.3. *Suppose $F\colon \mathcal{E} \to \mathcal{E}$ is a Hölder-continuous linear cocycle over a transitive C^2 Anosov diffeomorphism f and the fibers of $\mathcal{E}$ are two-dimensional. If for each periodic point $p \in \mathcal{M}$, the return map $F_p^n\colon \mathcal{E}_p \to \mathcal{E}_p$ is diagonalizable over $\mathbb{C}$ and its eigenvalues are equal in modulus, then F is conformal with respect to a Hölder-continuous Riemannian metric on $\mathcal{E}$.*

Moreover, if for each periodic point $p \in \mathcal{M}$, the return map $F_p^n\colon \mathcal{E}_p \to \mathcal{E}_p$ is diagonalizable over $\mathbb{C}$ and its eigenvalues are of modulus 1, then F is isometric with respect to a Hölder-continuous Riemannian metric on $\mathcal{E}$.

The proof of this result overcomes essential difficulties and substantially differs from the proof of Theorem 1.1. We use Zimmer's Amenable Reduction Theorem to recast the problem as one of continuity of measurable invariant conformal structures and of measurable invariant subbundles. We note that such

results on continuity of measurable invariant objects are rare beyond the case of group-valued functions with compact or abelian range.

In the next section we formulate our main technical results. In Section 3, we briefly introduce the main notions used in this paper. The proofs of all the results are given in Section 4.

2. Properties of cocycles

In this section we formulate our main technical results, which are of independent interest. We consider various conditions of relative pinching and establish properties of cocycles including existence and continuity of measurable invariant subbundles and conformal structures.

Standing assumptions. *In the statements below, f is a transitive C^2 Anosov diffeomorphism of a compact manifold $\mathcal{M}$, $P\colon \mathcal{E} \to \mathcal{M}$ is a finite-dimensional Hölder-continuous vector bundle over $\mathcal{M}$, and $F\colon \mathcal{E} \to \mathcal{E}$ is a Hölder-continuous linear cocycle over f with Hölder exponent β (see Section 3 for definitions).*

In the first proposition we obtain uniform relative pinching of the cocycle from asymptotic data at the periodic points. We denote by $\lambda_+(F, p)$ and $\lambda_-(F, p)$ the largest and smallest Lyapunov exponents of F at p, and by $\lambda_+(F, \mu)$ and $\lambda_-(F, \mu)$ the largest and smallest Lyapunov exponents of an ergodic invariant measure μ (see equations (3.3) and (3.4)).

Proposition 2.1. *Suppose there exists $\gamma \geq 0$ such that $\lambda_+(F, p) - \lambda_-(F, p) \leq \gamma$ for every f-periodic point $p \in \mathcal{M}$. Then, $\lambda_+(F, \mu) - \lambda_-(F, \mu) \leq \gamma$ for any ergodic invariant measure μ for f.*

Moreover, for any $\epsilon > 0$ there exists C_ϵ such that

$$(2.1) \qquad K_F(x, n) := \|F_x^n\| \cdot \left\|(F_x^n)^{-1}\right\| \leq C_\epsilon e^{(\gamma + \epsilon)|n|}$$

for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$.

We can apply this proposition to the case when at each periodic point p there is only one Lyapunov exponent, *i.e.,* all eigenvalues of $F_p^n \colon \mathcal{E}_p \to \mathcal{E}_p$ are of the same modulus. In this case we see that for any ergodic invariant measure for f, the cocycle F has only one Lyapunov exponent and (2.1) is satisfied with $\gamma = 0$.

In the proposition below, κ is the exponent in the Anosov condition (3.1) for f, and β is a Hölder exponent for F in (3.2). We show that under sufficient pinching, the iterates of the cocycle at the points on the same local stable manifold W_{loc}^s remain close. The same holds for the inverse map and the points on the same local unstable manifold W_{loc}^u. To consider the compositions $(F_x^n)^{-1} \circ F_y^n$ and $(F_x^{-n})^{-1} \circ F_y^{-n}$ we identify $\mathcal{E}_x$ and $\mathcal{E}_y$ for y close to x using local coordinates. This identification is Hölder.

Proposition 2.2. *Suppose that for some $0 < \epsilon < (\kappa\beta)/3$, there exists C_ϵ such that*

$$K_F(x, n) \leq C_\epsilon e^{\epsilon|n|} \qquad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

Then, there exist $C > 0$ and $\delta_0 > 0$ such that the following statements hold for any $\delta < \delta_0$ and $n \in \mathbb{N}$.

(a) *For any $x \in \mathscr{M}$ and $y \in W^s_{loc}(x)$ with $\mathrm{dist}(x, y) \le \delta$, we have*

$$\left\| (F^n_x)^{-1} \circ F^n_y - \mathrm{Id} \right\| \le C\delta^\beta.$$

(b) *For any $x \in \mathscr{M}$ and $y \in W^u_{loc}(x)$ with $\mathrm{dist}(x, y) \le \delta$, we have*

$$\left\| (F^{-n}_x)^{-1} \circ F^{-n}_y - \mathrm{Id} \right\| \le C\delta^\beta.$$

Next we establish continuity of measurable invariant conformal structures and subbundles. In our statements, we consider ergodic f-invariant measures on $\mathscr{M}$ with full support and local product structure. Examples include the measure of maximal entropy, and more generally Gibbs (equilibrium) measures of Hölder-continuous potentials. A measure μ has local product structure if it is locally equivalent to the product of its conditional measures on the local stable and unstable manifolds.

PROPOSITION 2.3. *Suppose that F satisfies the conclusion of Proposition 2.2, and μ is an ergodic f-invariant measure on $\mathscr{M}$ with full support and local product structure. Then any F-invariant measurable conformal structure on $\mathscr{E}$ defined μ almost everywhere is Hölder-continuous with exponent β.*

It is not known in general whether any measurable invariant conformal structure is continuous. Some results were established when the conformal structure is bounded [20] or belongs to L^p for sufficiently large p [13].

Combining Propositions 2.1, 2.2, and 2.3 we see that if at each periodic point there is only one Lyapunov exponent, or if the largest and the smallest exponents are sufficiently close, then any F-invariant measurable conformal structure on $\mathscr{E}$ is Hölder-continuous.

We recall that a cocycle F is said to be uniformly quasiconformal if the quasiconformal distortion $K_F(x, n)$ is uniformly bounded for all $x \in \mathscr{M}$ and $n \in \mathbb{Z}$, see Section 3.5 for details. In the next proposition we apply observations made by D. Sullivan [22] and P. Tukia [23] for quasiconformal group actions to our case. We state this result in greater generality than our standing assumptions. We note that the converse statement is also true.

PROPOSITION 2.4. *Let f be a diffeomorphism of a compact manifold $\mathscr{M}$ and let $F \colon \mathscr{E} \to \mathscr{E}$ be a continuous linear cocycle over f. If F is uniformly quasiconformal, then it preserves a bounded measurable conformal structure τ on $\mathscr{E}$.*

Under our standing assumptions, Proposition 2.3 implies that τ is Hölder-continuous. We can normalize it by a Hölder-continuous function on $\mathscr{M}$ to obtain a Riemannian metric with respect to which F is conformal, which yields the following corollary.

COROLLARY 2.5. *If F is uniformly quasiconformal then it preserves a Hölder-continuous conformal structure on $\mathscr{E}$, equivalently, F is conformal with respect to a Hölder-continuous Riemannian metric on $\mathscr{E}$.*

The above corollary, together with Propositions 2.1 and 2.2, enable us to prove Theorem 1.1.

Now we address continuity of measurable invariant subbundles. Note that the assumptions in the next proposition are stronger than those in Proposition 2.3. However, they are satisfied if at each periodic point there is only one Lyapunov exponent.

PROPOSITION 2.6. *Suppose that for any $\epsilon > 0$, there exists C_ϵ such that*

$$K_F(x, n) \leq C_\epsilon e^{\epsilon |n|} \qquad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

Then, any measurable F-invariant subbundle in $\mathcal{E}$ defined almost everywhere with respect to a measure with local product structure and full support is Hölder-continuous.

Combining Propositions 2.3 and 2.6 with Zimmer's Amenable Reduction Theorem we obtain the following description of cocycles with slowly growing quasiconformal distortion. We use it in the proof of Theorem 1.3.

PROPOSITION 2.7. *Suppose that for any $\epsilon > 0$, there exists C_ϵ such that*

$$K_F(x, n) \leq C_\epsilon e^{\epsilon |n|} \qquad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

Then, either F preserves a Hölder-continuous conformal structure on $\mathcal{E}$, or F preserves a Hölder-continuous proper nontrivial subbundle $\mathcal{E}'$ of $\mathcal{E}$ and a Hölder-continuous conformal structure on $\mathcal{E}'$.

Note that the alternatives are not mutually exclusive. If $\mathcal{E}'$ is one-dimensional then having a conformal structure on it becomes trivial.

3. PRELIMINARIES

In this section, we briefly introduce the main notions used in this paper.

3.1. **Anosov diffeomorphisms.** Let f be a diffeomorphism of a compact Riemannian manifold $\mathcal{M}$. It is said to be Anosov if there exists a decomposition of the tangent bundle $T\mathcal{M}$ into two invariant continuous subbundles E^s and E^u, and there exists constants $C > 0$, $\kappa > 0$ such that for all $n \in \mathbb{N}$,

$$(3.1) \qquad Ce^{-\kappa n} \|v\| \geq \begin{cases} \|df^n(v)\| & \text{for all } v \in E^s, \\ \|df^{-n}(v)\| & \text{for all } v \in E^u. \end{cases}$$

The distributions E^s and E^u are called stable and unstable. These distributions are tangential to the foliations W^s and W^u respectively. The local stable and unstable leaves $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(x)$ are the connected components of x in the intersection of $W^s(x)$ and $W^u(x)$ with a small ball around x.

3.2. **Hölder-continuous vector bundles.** Let $\mathscr{M}$ be a compact smooth manifold. We consider a finite-dimensional Hölder-continuous vector bundle $P\colon \mathscr{E} \to \mathscr{M}$ over $\mathscr{M}$. By this we mean that there exists an open cover $\{U_i\}$ of $\mathscr{M}$ and a system of local coordinates $\phi_i\colon P^{-1}(U_i) \to U_i \times \mathbb{R}^d$ such that the coordinate changes

$$\phi_j \circ \phi_i^{-1}\colon (U_i \cap U_j) \times \mathbb{R}^d \to (U_i \cap U_j) \times \mathbb{R}^d$$
$$(x, v) \mapsto \big(x, L_x(v)\big)$$

are homeomorphisms with linear automorphisms L_x depending in a Hölder-continuous way on x. That is, there exist $C, \beta > 0$ such that

$$\|L_x - L_y\| \le C \operatorname{dist}(x, y)^{\beta}$$

for all i, j and all $x, y \in U_i \cap U_j$.

We will sometimes identify the fibers at nearby points using the local coordinates. We equip $\mathscr{E}$ with a background Hölder-continuous Riemannian metric, *i.e.,* a family of inner products on the fibers $\mathscr{E}_x$ depending Hölder-continuously on x.

3.3. **Linear cocycles and Lyapunov exponents.** Let f be a diffeomorphism of a compact smooth manifold $\mathscr{M}$ and $P\colon \mathscr{E} \to \mathscr{M}$ be a finite-dimensional Hölder-continuous vector bundle over $\mathscr{M}$. A Hölder-continuous linear cocycle over f is a homeomorphism $F\colon \mathscr{E} \to \mathscr{E}$ such that $P \circ F = f \circ P$ and $F_x\colon \mathscr{E}_x \to \mathscr{E}_{f(x)}$ is a linear isomorphism which depends Hölder-continuously on x, *i.e.,* there exist $C, \beta > 0$ such that for all nearby $x, y \in \mathscr{M}$,

$$(3.2) \qquad \|F_x - F_y\| + \big\|F_x^{-1} - F_y^{-1}\big\| \le C \operatorname{dist}(x, y)^{\beta}.$$

Here F_x and F_y are viewed as matrices using local coordinates. Note that the second term on the left is not necessary for a continuous F. Indeed, F_x^{-1} is then automatically continuous in x and bounded on $\mathscr{M}$, so we can estimate

$$\big\|F_x^{-1} - F_y^{-1}\big\| = \big\|F_x^{-1}(F_y - F_x)F_y^{-1}\big\| \le C'\|F_x - F_y\|.$$

We consider the standard notion of Lyapunov exponents for such a cocycle F (see [3, Section 2.3] for more details). We emphasize that the Lyapunov exponents of F are defined for vectors in the linear spaces $\mathscr{E}_x$. Note that for any measure μ on $\mathscr{M}$ the vector bundle $\mathscr{E}$ is trivial on a set of full measure. By Oseledets's Multiplicative Ergodic Theorem the Lyapunov exponents of F, as well as Lyapunov decomposition of $\mathscr{E}$, are defined almost everywhere for every ergodic f-invariant measure μ on $\mathscr{M}$. In particular, they are defined at every periodic point. We are primarily interested in the largest and the smallest Lyapunov exponents of μ which can be defined, respectively, as follows:

$$(3.3) \quad \lambda_+(F, \mu) = \lambda_+(F, x) = \lim_{n\to\infty} \frac{1}{n} \log \|F_x^n\| \quad \text{for } \mu\text{-almost every } x \in \mathscr{M},$$

$$(3.4) \quad \lambda_-(F, \mu) = \lambda_-(F, x) = \lim_{n\to\infty} \frac{1}{n} \log \big\|(F_x^n)^{-1}\big\|^{-1} \quad \text{for } \mu\text{-almost every } x \in \mathscr{M},$$

where $F_x^n = F_{f^{n-1}x} \circ \cdots \circ F_{f(x)} \circ F_x$.

3.4. **Conformal structures.** A conformal structure on $\mathbb{R}^d$, $d \geq 2$, is a class of proportional inner products. The space $\mathscr{C}^d$ of conformal structures on $\mathbb{R}^d$ identifies with the space of real symmetric positive-definite $d \times d$ matrices with determinant 1, which is isomorphic to $SL(d, \mathbb{R})/SO(d, \mathbb{R})$. The group $GL(d, \mathbb{R})$ acts transitively on $\mathscr{C}^d$ via

$$X[C] = \left(\det X^T X\right)^{-1/d} X^T C X, \qquad \text{where } X \in GL(d, \mathbb{R}) \text{ and } C \in \mathscr{C}^d.$$

It is known that $\mathscr{C}^d$ becomes a Riemannian symmetric space of nonpositive curvature when equipped with a certain $GL(d, \mathbb{R})$-invariant metric. The distance to the identity in this metric is given by

$$\text{dist}(\text{Id}, C) = \left(\sqrt{d}/2\right)\left((\log \lambda_1)^2 + \cdots + (\log \lambda_d)^2\right)^{1/2},$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of C (see [23, p.327] for more details and [16, p.27] for the formula). The distance between two structures C_1 and C_2 can be computed as $\text{dist}(C_1, C_2) = \text{dist}(\text{Id}, X[C_2])$, where $X[C_1] = \text{Id}$.

It is easy to check the following relation between this metric and the operator norm

$$(3.5) \qquad \left(\sqrt{d}/8\right)\log\left(\|C\| \cdot \|C^{-1}\|\right) \leq \text{dist}(\text{Id}, C) \leq (d/2)\max\{\log\|C\|, \log\|C^{-1}\|\}.$$

We also note that $\|C^{-1}\| \leq \|C\|^{d-1}$. Thus, a subset of $\mathscr{C}^d$ is bounded with respect to this distance if and only if it is bounded with respect to the operator norm. We also note that on any bounded subset of $\mathscr{C}^d$ this distance is bi-Lipschitz equivalent to the distance induced by the operator norm on matrices.

Let $\mathscr{E}$ be a Hölder-continuous vector bundle over a compact manifold $\mathscr{M}$. A conformal structure on $\mathscr{E}_x$ is a class of proportional inner products on $\mathscr{E}_x$. Using the background Riemannian metric on $\mathscr{E}$, we can identify an inner product with a symmetric linear operator with determinant 1 as before. For each $x \in \mathscr{M}$, we denote the space of conformal structures on $\mathscr{E}_x$ by $\mathscr{C}(x)$. Thus we obtain a bundle $\mathscr{C}$ over $\mathscr{M}$ whose fiber over x is $\mathscr{C}(x)$. We equip the fibers of $\mathscr{C}$ with the Riemannian metric defined above. A continuous (Hölder-continuous, measurable) section of $\mathscr{C}$ is called a continuous (Hölder-continuous, measurable) conformal structure on $\mathscr{E}$. A measurable conformal structure τ on $\mathscr{E}$ is said to be *bounded* if the distance between $\tau(x)$ and $\tau_0(x)$ is uniformly bounded on $\mathscr{M}$ for a continuous conformal structure τ_0 on $\mathscr{E}$.

Now, let f be a diffeomorphism of $\mathscr{M}$ and $F: \mathscr{E} \to \mathscr{E}$ be a linear cocycle over f. Then F induces a natural pullback action F^* on conformal structures as follows. For a conformal structure $\tau(f(x)) \in \mathscr{C}(f(x))$, viewed as the linear operator on $\mathscr{E}_{f(x)}$, $F_x^*(\tau(f(x))) \in \mathscr{C}(x)$ is given by

$$(3.6) \qquad F_x^*\left(\tau(f(x))\right) = \left(\det\left((F_x)^T \circ F_x\right)\right)^{-1/n} (F_x)^T \circ \tau(f(x)) \circ F_x,$$

where $(F_x)^T: \mathscr{E}_{f(x)} \to \mathscr{E}_x$ denotes the conjugate operator of F_x. We note that $F_x^*: \mathscr{C}(f(x)) \to \mathscr{C}(x)$ is an isometry between the fibers $\mathscr{C}(f(x))$ and $\mathscr{C}(x)$.

We say that a conformal structure τ is *F-invariant* if $F^*(\tau) = \tau$.

3.5. **Uniform quasiconformality.** Let f be a diffeomorphism of a compact manifold $\mathcal{M}$ and $F: \mathcal{E} \to \mathcal{E}$ be a linear cocycle over f. For $x \in \mathcal{M}$ and $n \in \mathbb{Z}$ the *quasiconformal distortion* of F is defined by

$$K_F(x, n) := \frac{\max\{\|F_x^n(v)\| : v \in \mathcal{E}_x, \|v\| = 1\}}{\min\{\|F_x^n(v)\| : v \in \mathcal{E}_x, \|v\| = 1\}} = \|F_x^n\| \cdot \|(F_x^n)^{-1}\|.$$

We say that F is *uniformly quasiconformal* if $K_F(x, n)$ is uniformly bounded for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$. If $K_F(x, n) = 1$ for all x and n, then F is said to be *conformal*.

Clearly, F is conformal with respect to a Riemannian metric on $\mathcal{E}$ if and only if it preserves the conformal structure associated with this metric. We note that the notion of uniform quasiconformality does not depend on the choice of a continuous metric. So if F preserves a continuous conformal structure on $\mathcal{E}$ then F is uniformly quasiconformal on $\mathcal{E}$ with respect to any continuous metric on $\mathcal{E}$. Corollary 2.5 shows that the converse is also true if f is a transitive Anosov diffeomorphism.

4. PROOFS

4.1. **Proof of Proposition 2.1.** To show that $\lambda_+(\mu) - \lambda_-(\mu) \le \gamma$ for any ergodic invariant measure μ for f, we apply the following theorem.

THEOREM 4.1 ([6, Theorem 1.4])**.** *Let f be a homeomorphism of a compact metric space X satisfying the closing property, let F be a Hölder-continuous $GL(d, \mathbb{R})$ cocycle over f, and let μ be an ergodic invariant measure for f. Then the Lyapunov exponents $\lambda_1 \le \cdots \le \lambda_d$ (listed with multiplicities) of F with respect to μ can be approximated by the Lyapunov exponents of F at periodic points. More precisely, for any $\epsilon > 0$ there exists a periodic point $p \in X$ for which the Lyapunov exponents $\lambda_1^{(p)} \le \cdots \le \lambda_d^{(p)}$ of F satisfy $|\lambda_i - \lambda_i^{(p)}| < \epsilon$ for $i = 1, \ldots, d$.*

As stated in the remark after this theorem, it holds for any Hölder-continuous linear cocycle F. Also, a transitive Anosov diffeomorphism satisfies the closing property. Thus we can apply the theorem in our context and immediately obtain the desired result for μ.

Now we prove the estimate for the quasiconformal distortion $K_F(x, n)$ using the following result.

PROPOSITION 4.2 ([19, Proposition 3.4])**.** *Let $f: \mathcal{M} \to \mathcal{M}$ be a continuous map of a compact metric space. Let $a_n: \mathcal{M} \to \mathbb{R}$, $n \ge 0$ be a sequence of continuous functions such that*

(4.1) $\qquad a_{n+k}(x) \le a_n\big(f^k(x)\big) + a_k(x) \qquad$ *for every $x \in \mathcal{M}$, $n, k \ge 0$,*

and such that there is a sequence of continuous functions b_n, $n \ge 0$, satisfying

(4.2) $\qquad a_n(x) \le a_n\big(f^k(x)\big) + a_k(x) + b_k\big(f^n(x)\big) \qquad$ *for every $x \in \mathcal{M}$, $n, k \ge 0$.*

If $\inf_n\big(\frac{1}{n} \int_{\mathcal{M}} a_n \, d\mu\big) < 0$ for every ergodic f-invariant measure, then there is $N \ge 0$ such that $a_N(x) < 0$ for every $x \in \mathcal{M}$.

To simplify the notations, we write $K(x,n)$ for $K_F(x,n)$. For a given $\epsilon > 0$ we apply Proposition 4.2 to

$$a_n(x) = \log K(x,n) - (\gamma + \epsilon)n \quad \text{and} \quad b_n(x) = \log K(x,n) + (\gamma + \epsilon)n.$$

It is easy to see from the definition of the quasiconformal distortion that

$$K(x,n+k) \le K(x,k) \cdot K\big(f^k(x),n\big) \quad \text{and} \quad K(x,n+k) \ge K(x,n) \cdot \big[K\big(f^n(x),k\big)\big]^{-1}$$

for every $x \in \mathcal{M}$ and $n,k \ge 0$. It follows that $a_{n+k}(x) \le a_n(f^k(x)) + a_k(x)$, i.e., the functions a_n satisfy (4.1), and $a_{n+k}(x) \ge a_n(x) - b_k(f^n(x))$. Hence

$$a_n(x) \le a_{n+k}(x) + b_k\big(f^n(x)\big) \le a_n\big(f^k(x)\big) + a_k(x) + b_k\big(f^n(x)\big)$$

and we obtain (4.2).

Let μ be an ergodic f-invariant measure. We note that since a_n satisfy (4.1), the Subadditive Ergodic Theorem implies that

$$\inf_n \frac{1}{n} \int_{\mathcal{M}} a_n \, d\mu = \lim_{n \to \infty} \frac{1}{n} a_n(x) \quad \text{for } \mu \text{ almost all } x \in \mathcal{M}.$$

Using the definitions of $K(x,n)$, $\lambda_+(F,\mu)$ and $\lambda_-(F,\mu)$, we obtain that for μ almost all x

$$\begin{aligned}
\lim_{n \to \infty} \frac{\log K(x,n)}{n} &= \lim_{n \to \infty} \frac{1}{n} \log(\|F_x^n\| \cdot \|(F_x^n)^{-1}\|) \\
&= \lim_{n \to \infty} \frac{1}{n} \log \|F_x^n\| - \lim_{n \to \infty} \frac{1}{n} \log \|(F_x^n)^{-1}\|^{-1} \\
&= \lambda_+(F,\mu) - \lambda_-(F,\mu) \\
&\le \gamma.
\end{aligned}$$

Hence, $\lim_{n \to \infty} \frac{1}{n} a_n(x) \le -\epsilon < 0$ for μ almost all $x \in \mathcal{M}$.

Thus, all assumptions of Proposition 4.2 are satisfied. Hence for any $\epsilon > 0$ there exists N_ϵ such that $a_{N_\epsilon}(x) < 0$, i.e., $K(x,N_\epsilon) \le e^{(\gamma+\epsilon)N_\epsilon}$ for all $x \in \mathcal{M}$. For any $n > 0$, we write $n = mN_\epsilon + r$, $0 \le r < N_\epsilon$, and estimate

$$\begin{aligned}
K(x,n) &\le K(x,r) \cdot K(f^r(x),N_\epsilon) \cdot K(f^{r+N_\epsilon} x, N_\epsilon) \cdots K(f^{r+(m-1)N_\epsilon} x, N_\epsilon) \\
&\le K(x,r) \cdot e^{(\gamma+\epsilon)mN_\epsilon} \\
&\le C_\epsilon e^{(\gamma+\epsilon)n},
\end{aligned}$$

where $C_\epsilon = \max K(x,r)$ with the maximum taken over all $x \in \mathcal{M}$ and $1 \le r < N_\epsilon$. Since $K(x,n) = K(f^n x, -n)$ we obtain $K(x,n) \le C_\epsilon e^{(\gamma+\epsilon)|n|}$ for all x in $\mathcal{M}$ and n in $\mathbb{Z}$. $\square$

4.2. **Proof of Proposition 2.2.** First consider the case when $y \in W_{\text{loc}}^s(x)$. Since at least one of the points x and y is nonperiodic, we assume that x is. We write

$x_i = f^i(x)$ and $y_i = f^i(y)$ for $i = 0, 1, \ldots, n$. We have

$$
\begin{aligned}
(F_x^n)^{-1} \circ F_y^n &= (F_x^{n-1})^{-1} \circ \big((F_{x_{n-1}})^{-1} \circ F_{y_{n-1}}\big) \circ F_y^{n-1} \\
&= (F_x^{n-1})^{-1} \circ (\mathrm{Id} + r_{n-1}) \circ F_y^{n-1} \\
&= (F_x^{n-1})^{-1} \circ F_y^{n-1} + (F_x^{n-1})^{-1} \circ r_{n-1} \circ F_y^{n-1} \\
&= \cdots = \mathrm{Id} + \sum_{i=0}^{n-1} (F_x^i)^{-1} \circ r_i \circ F_y^i,
\end{aligned}
$$

where $(F_{x_i})^{-1} \circ F_{y_i} = \mathrm{Id} + r_i$. We estimate

$$
(4.3) \qquad \big\| \mathrm{Id} - (F_x^n)^{-1} \circ F_y^n \big\| \le \sum_{i=0}^{n-1} \big\| (F_x^i)^{-1} \big\| \cdot \| r_i \| \cdot \| F_y^i \|.
$$

Since F is Hölder-continuous with exponent β, we have

$$
\| r_i \| = \big\| (F_{x_i})^{-1} \circ F_{y_i} - \mathrm{Id} \big\| \le \big\| (F_{x_i})^{-1} \big\| \cdot \| F_{y_i} - F_{x_i} \| \le C_0 \operatorname{dist}(x_i, y_i)^{\beta}.
$$

Since $y \in W_{\mathrm{loc}}^s(x)$, for κ as in (3.1) we obtain

$$
(4.4) \qquad \| r_i \| \le C_0 \big(C_1 \operatorname{dist}(x, y) e^{-\kappa i} \big)^{\beta} \le C_0 \big(C_1 \delta e^{-\kappa i} \big)^{\beta} \le C_2 \delta^{\beta} e^{-\kappa \beta i}.
$$

Lemma 4.4 below shows that

$$
(4.5) \qquad \big\| (F_x^i)^{-1} \big\| \cdot \| F_y^i \| \le C_3 e^{3i\epsilon} \qquad \text{for } i = 0, \ldots, n-1.
$$

Combining (4.3), (4.4), and (4.5), we obtain

$$
\big\| \mathrm{Id} - (F_x^n)^{-1} \circ F_y^n \big\| \le \sum_{i=0}^{n-1} \big(C_2 \delta^{\beta} e^{-\kappa \beta i} \big) \big(C_3 e^{3i\epsilon} \big) \le C_2 C_3 \delta^{\beta} \sum_{i=0}^{n-1} \big(e^{3\epsilon - \kappa\beta} \big)^i \le C \delta^{\beta},
$$

since $3\epsilon - \kappa\beta < 0$. This completes the proof for the case of $y \in W^s(x)$.

To prove (b), we observe that F^{-1} satisfies the assumptions of the proposition. Indeed, $K_{F^{-1}}(x, n) = \| (F_x^{-n}) \| \cdot \| (F_x^{-n})^{-1} \| = K_F(x, n)$. Thus we can apply (a) to F^{-1}, which yields (b).

It remains to prove estimate (4.5). To do this, we construct special metrics on $\mathcal{E}_{f^k x}$ along the orbit of a nonperiodic point $x \in \mathcal{M}$. We denote $x_k = f^k(x)$, $k \in \mathbb{Z}$.

LEMMA 4.3. *Let f be a diffeomorphism of a compact manifold $\mathcal{M}$, $\mathcal{E}$ be a continuous vector bundle over $\mathcal{M}$, and F be a continuous linear cocycle over f. Suppose that for some $\epsilon > 0$ there exists C_ϵ such that $K_F(x, n) \le C_\epsilon e^{\epsilon|n|}$ for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$. Then for any nonperiodic point $x \in \mathcal{M}$, there exist metrics $\| \cdot \|_{x_k}$ on $\mathcal{E}_{x_k}$, $k \in \mathbb{Z}$, such that*

$$
(4.6) \qquad \frac{\max\{ \| F_{x_k}^n(v) \|_{x_{k+n}} : v \in \mathcal{E}_{x_k}, \ \| v \|_{x_k} = 1 \}}{\min\{ \| F_{x_k}^n(v) \|_{x_{k+n}} : v \in \mathcal{E}_{x_k}, \ \| v \|_{x_k} = 1 \}} \le e^{3|n|\epsilon} \qquad \text{for all } k, n \in \mathbb{Z}.
$$

Moreover, there exists a constant M_ϵ such that $\| v \| \le \| v \|_{x_k} \le M_\epsilon \| v \|$ for all $k \in \mathbb{Z}$ and $v \in \mathcal{E}_{x_k}$, where $\| \cdot \|$ is a given continuous metric on $\mathcal{E}$.

Proof. We choose a unit vector $u \in \mathscr{E}_x$ and set $u_k = F_x^k(u)/\|F_x^k(u)\| \in \mathscr{E}_{x_k}$. For a vector $v \in \mathscr{E}_{x_k}$ we define

$$\|v\|_{x_k}^2 = \sum_{m=-\infty}^{\infty} \frac{\|F_{x_k}^m(v)\|^2}{\|F_{x_k}^m(u_k)\|^2 \cdot e^{3|m|\epsilon}}.$$

By the assumption on K_F, $\|F_{x_k}^m(\frac{v}{\|v\|})\| \cdot \|F_{x_k}^m(u_k)\|^{-1} \le C_\epsilon e^{|m|\epsilon}$ and hence the terms of this series are bounded by $C_\epsilon^2 e^{-|m|\epsilon}\|v\|^2$. This implies that the series converges and $\|v\|_{f^k x}^2 \le M_\epsilon^2 \|v\|^2$, where $M_\epsilon^2 = C_\epsilon^2 \sum_{n=-\infty}^{\infty} e^{-|m|\epsilon}$. Clearly, $\|v\|_{x_k}^2$ is at least the term with $m = 0$, and thus $\|v\|_{x_k} \ge \|v\|$.

We note that it suffices to prove the estimate for $n = 1$, then it automatically follows for all n. We observe that u_{k+1} is a unit vector parallel to $F_{x_k}(u_k)$, and hence $u_{k+1} = F_{x_k}(u_k)/\|F_{x_k}(u_k)\|$. For any vector $v \in \mathscr{E}_{x_k}$ we estimate

$$\begin{aligned}
\|F_{x_k}(v)\|_{x_{k+1}}^2 &= \sum_{m=-\infty}^{\infty} \frac{\|F_{x_{k+1}}^m(F_{x_k}(v))\|^2}{\|F_{x_{k+1}}^m(u_{k+1})\|^2 \cdot e^{3|m|\epsilon}} \\
&= \sum_{m=-\infty}^{\infty} \frac{\|F_{x_{k+1}}^m(F_{x_k}(v))\|^2 \cdot \|F_{x_k}(u_k)\|^2}{\|F_{x_{k+1}}^m(F_{x_k}(u_k))\|^2 \cdot e^{3|m|\epsilon}} \\
&= \sum_{m=-\infty}^{\infty} \frac{\|F_{x_k}^{m+1}(v)\|^2 \cdot \|F_{x_k}(u_k)\|^2}{\|F_{x_k}^{m+1}(u_k)\|^2 \cdot e^{3|m|\epsilon}} \\
&= \|F_{x_k}(u_k)\|^2 \sum_{j=-\infty}^{\infty} \frac{\|F_{x_k}^j(v)\|^2}{\|F_{x_k}^j(u_k)\|^2 \cdot e^{3|j-1|\epsilon}} \\
&\le \|F_{x_k}(u_k)\|^2 \sum_{j=-\infty}^{\infty} \frac{\|F_{x_k}^j(v)\|^2 \cdot e^{3\epsilon}}{\|F_{x_k}^j(u_k)\|^2 \cdot e^{3|j|\epsilon}} \\
&\le \|F_{x_k}(u_k)\|^2 \cdot \|v\|_{x_k}^2 \cdot e^{3\epsilon}.
\end{aligned}$$

Here we used the estimate $|j| - 1 \le |j - 1|$. Similarly, using $|j - 1| \le |j| + 1$, we obtain $\|F_{x_k}(v)\|_{x_{k+1}}^2 \ge \|F_{x_k}(u_k)\|^2 \cdot \|v\|_{x_k}^2 \cdot e^{-3\epsilon}$. Thus, for any vector $v \in \mathscr{E}_{x_k}$

$$e^{-(3/2)\epsilon} \cdot \|F_{x_k}(u_k)\| \cdot \|v\|_{x_k} \le \|F_{x_k}(v)\|_{x_{k+1}} \le e^{(3/2)\epsilon} \cdot \|F_{x_k}(u_k)\| \cdot \|v\|_{x_k}.$$

It follows that for any two vectors $v, w \in \mathscr{E}_{x_k}$ with $\|v\|_{x_k} = \|w\|_{x_k} = 1$

$$e^{-3\epsilon}\|F_{x_k}(v)\|_{x_{k+1}} \le \|F_{x_k}(w)\|_{x_{k+1}} \le e^{3\epsilon}\|F_{x_k}(v)\|_{x_{k+1}}.$$

Hence, $e^{-3\epsilon} \le \|F_{x_k}(w)\|_{x_{k+1}}/\|F_{x_k}(v)\|_{x_{k+1}} \le e^{3\epsilon}$. $\square$

LEMMA 4.4. *For $i = 0, \dots, n-1$, $\|(F_x^i)^{-1}\| \cdot \|F_y^i\| \le C_3 e^{3i\epsilon}$.*

Proof. Consider the metrics $\|\cdot\|_{x_k}$ on $\mathscr{E}_{x_k}$, $k = 0, \dots, i$, given by Lemma 4.3. We denote by $\|F_{x_k}\|_k$ the norm of the operator F from $(\mathscr{E}_{x_k}, \|\cdot\|_{x_k})$ to $(\mathscr{E}_{x_{k+1}}, \|\cdot\|_{x_{k+1}})$ and we denote by $\|(F_{x_k})^{-1}\|_k$ the norm of the corresponding inverse operator. Since $\|v\| \le \|v\|_{x_k} \le M_\epsilon \|v\|$ for any $v \in \mathscr{E}_{x_k}$, it is easy to see that $(1/M_\epsilon)\|F_{x_k}\| \le$

$\|F_{x_k}\|_k \le M_\epsilon \|F_{x_k}\|$. Using Hölder-continuity of F in the metric $\|\cdot\|$, we obtain

$$
\begin{aligned}
\frac{\|F_{x_k}\|_k}{\|F_{y_k}\|_k} &\le 1 + \frac{\left|\|F_{x_k}\|_k - \|F_{y_k}\|_k\right|}{\|F_{y_k}\|_k} \\
&\le 1 + \frac{\|F_{x_k} - F_{y_k}\|_k}{\|F_{y_k}\|_k} \\
&\le 1 + \frac{M_\epsilon^2 \|F_{x_k} - F_{y_k}\|}{\|F_{y_k}\|} \\
&\le 1 + \frac{M_\epsilon^2 c_1\big(\mathrm{dist}(x_k, y_k)\big)^\beta}{\min_z \|F_z\|} \\
&= 1 + c_2\big(\mathrm{dist}(x_k, y_k)\big)^\beta.
\end{aligned}
$$

For $n = 1$, inequality (4.6) gives $\|F_{x_k}\|_k \cdot \|(F_{x_k})^{-1}\|_k \le e^{3\epsilon}$, and we estimate

$$
\begin{aligned}
\|(F_x^i)^{-1}\|_i &\cdot \|F_y^i\|_i \\
&\le \|(F_x)^{-1}\|_0 \cdot \|(F_{x_1})^{-1}\|_1 \cdots \|(F_{x_{i-1}})^{-1}\|_{i-1} \cdot \|F_y\|_0 \cdot \|F_{y_1}\|_1 \cdots \|F_{y_{i-1}}\|_{i-1} \\
&\le \frac{\|F_y\|_0}{\|F_x\|_0} e^{3\epsilon} \cdot \frac{\|F_{y_1}\|_1}{\|F_{x_1}\|_1} e^{3\epsilon} \cdots \frac{\|F_{y_{i-1}}\|_{i-1}}{\|F_{x_{i-1}}\|_{i-1}} e^{3\epsilon} \\
&\le e^{3i\epsilon} \prod_{k=0}^{i-1} \left(1 + c_2\big(\mathrm{dist}(x_k, y_k)\big)^\beta\right) \\
&\le e^{3i\epsilon} \prod_{k=0}^{i-1} \left(1 + c_2\big(C_1 \delta e^{-\kappa k}\big)^\beta\right) \\
&\le e^{3i\epsilon} \prod_{k=0}^{i-1} \left(1 + c_3 e^{-\beta\kappa k}\right) \\
&\le c_4 e^{3i\epsilon}.
\end{aligned}
$$

It follows that $\|(F_x^i)^{-1}\| \cdot \|F_y^i\| \le M_\epsilon^2 c_4 e^{3i\epsilon} = C_3 e^{3i\epsilon}$. $\qquad\square$

This completes the proof of Proposition 2.2.

4.3. **Proof of Proposition 2.3.** We identify the spaces of conformal structures at nearby points by identifying the fibers of $\mathscr{E}$ with $\mathbb{R}^d$ using local coordinates. We use the distance between conformal structures described in Section 3.4. Let τ be an invariant μ-measurable conformal structure on $\mathscr{E}$.

First we estimate the distance between the values of τ at x and at a nearby point $y \in W_{\mathrm{loc}}^s(x)$. Let $x_n = f^n(x)$, $y_n = f^n(y)$, and let $D_x := (F_x^n)^*$ be the isometry from $\mathscr{C}(f^n x)$ to $\mathscr{C}(x)$ induced by F_x^n (see (3.6)). Since the conformal structure τ is invariant, $\tau(x) = D_x(\tau(x_n))$ and $\tau(y) = D_y(\tau(y_n))$. Using this and the

fact that D_y is an isometry, we obtain

$$
\begin{aligned}
\mathrm{dist}(\tau(x), \tau(y)) &= \mathrm{dist}\big(D_x(\tau(x_n)), D_y(\tau(y_n))\big) \\
&\le \mathrm{dist}\big(D_x(\tau(x_n)), D_y(\tau(x_n))\big) + \mathrm{dist}\big(D_y(\tau(x_n)), D_y(\tau(y_n))\big) \\
&= \mathrm{dist}\big(\tau(x_n), ((D_x)^{-1} \circ D_y)(\tau(x_n))\big) + \mathrm{dist}(\tau(x_n), \tau(y_n)).
\end{aligned}
$$

To estimate $\mathrm{dist}\big(\tau(x_n), ((D_x)^{-1} \circ D_y)(\tau(x_n))\big)$, we use the following lemma.

LEMMA 4.5. *Let σ be a conformal structure on $\mathbb{R}^d$ and let A be a linear transformation of $\mathbb{R}^d$ sufficiently close to identity. Then,*

$$
\mathrm{dist}\big(\sigma, A^*(\sigma)\big) \le k(\sigma) \|A - \mathrm{Id}\|,
$$

where $k(\sigma)$ is bounded on compact sets in $\mathscr{C}^d$. More precisely, if σ is given by a matrix C, then $k(\sigma) \le 3d\|C^{-1}\| \cdot \|C\|$ for any A with $\|A - \mathrm{Id}\| \le \big(6\|C^{-1}\| \cdot \|C\|\big)^{-1}$.

Proof. We write $A = \mathrm{Id} + R$. Recall that the matrix C corresponding to σ is symmetric and positive-definite with determinant 1. Thus there exists an orthogonal matrix Q such that $Q^T C Q$ is a diagonal matrix whose diagonal entries are the eigenvalues $\lambda_i > 0$ of C. Let X be the product of Q and the diagonal matrix with entries $1/\sqrt{\lambda_i}$. Then X has determinant 1 and $X[C] = X^T C X = \mathrm{Id}$. Now we estimate

$$
\begin{aligned}
\mathrm{dist}\big(\sigma, A^*(\sigma)\big) = \mathrm{dist}(C, A[C]) &= \mathrm{dist}(\mathrm{Id}, X[A[C]]) \\
&= \mathrm{dist}\big(\mathrm{Id}, X^T A^T C A X\big) \\
&= \mathrm{dist}\big(\mathrm{Id}, X^T (\mathrm{Id} + R^T) C (\mathrm{Id} + R) X\big) \\
&= \mathrm{dist}(\mathrm{Id}, \mathrm{Id} + B),
\end{aligned}
$$

where

$$
B = X^T C R X + X^T R^T C X + X^T R^T C R X.
$$

Since $\|R\| \le 1$, we observe that $\|B\| \le 3\|X\|^2 \cdot \|C\| \cdot \|R\|$. Also, $\|X\|^2 \le \|C^{-1}\|$, as follows from the construction of X. Therefore, $\|B\| \le 3\|C^{-1}\| \cdot \|C\| \cdot \|R\|$. Since $\|R\| \le (6\|C^{-1}\| \cdot \|C\|)^{-1}$, it follows that $\|B\| \le 1/2$. Hence, $\|(\mathrm{Id} + B)^{-1}\| \le 1 + 2\|B\|$. Using (3.5), we estimate

$$
\begin{aligned}
\mathrm{dist}\big(\sigma, A^*(\sigma)\big) = \mathrm{dist}(\mathrm{Id}, \mathrm{Id} + B) &\le (d/2)\log\big(\max\{\|\mathrm{Id} + B\|, \|(\mathrm{Id} + B)^{-1}\|\}\big) \\
&\le (d/2)\log(1 + 2\|B\|) \\
&\le d\|B\| \\
&\le 3d\|C^{-1}\| \cdot \|C\| \cdot \|R\|. \qquad \square
\end{aligned}
$$

Since the conformal structure τ is μ-measurable, by Lusin's Theorem there exists a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ on which τ is uniformly continuous and bounded.

We now show that for x_n in S the term $\mathrm{dist}\big(\tau(x_n), ((D_x)^{-1} \circ D_y)(\tau(x_n))\big)$ is Hölder in $\mathrm{dist}(x, y)$. For this, we observe that the map $(D_x)^{-1} \circ D_y$ is induced by $(F_x^n)^{-1} \circ F_y^n$, and $\|(F_x^n)^{-1} \circ F_y^n - \mathrm{Id}\| \le k\,\mathrm{dist}(x, y)^\beta$ by the assumption. We apply

Lemma 4.5 to $\sigma = \tau(x_n)$ and $A = (F_x^n)^{-1} \circ F_y^n$. Since the conformal structure τ is bounded on S, so are $\|C^{-1}\|$ and $\|C\|$. We obtain that

$$\text{dist}\big(\tau(x_n), ((D_x)^{-1} \circ D_y)(\tau(x_n))\big) \le k\tau(x_n)\big\|(F_x^n)^{-1} \circ F_y^n - \text{Id}\big\| \le k_1 \text{dist}(x,y)^\beta,$$

where the constant k_1 depends on the set S. We conclude that if x_n is in S, then

$$\text{dist}(\tau(x), \tau(y)) \le \text{dist}(\tau(x_n), \tau(y_n)) + k_1 \text{dist}(x,y)^\beta.$$

Let G be the set of points in $\mathcal{M}$ for which the frequency of visiting S equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem $\mu(G) = 1$. If both x and y are in G, then there exists a sequence $\{n_i\}$ such that $x_{n_i} \in S$ and $y_{n_i} \in S$. Since $y \in W_{\text{loc}}^s(x)$, $\text{dist}(x_{n_i}, y_{n_i}) \to 0$ and hence $\text{dist}(\tau(x_{n_i}), \tau(y_{n_i})) \to 0$ by continuity of τ on S. Thus, we obtain

$$\text{dist}(\tau(x), \tau(y)) \le k^s \text{dist}(x,y)^\beta.$$

By a similar argument, we have $\text{dist}(\tau(x), \tau(z)) \le k^u \text{dist}(x,z)^\beta$ for $x, z \in G$ with $z \in W_{\text{loc}}^u(x)$.

Consider a small open set in $\mathcal{M}$ with a product structure. For μ almost all local stable leaves, the set of points of G on the leaf has full conditional measure. Consider points $x, y \in G$ lying on two such local stable leaves. We denote by $H_{x,y}$ be the unstable holonomy map between $W_{\text{loc}}^s(x)$ and $W_{\text{loc}}^s(y)$. Since μ has local product structure, the holonomy maps are absolutely continuous with respect to the conditional measures. Hence there exists a point $z \in W_{\text{loc}}^s(x) \cap G$ close to x such that $H_{x,y}(z)$ is also in G. By the above argument, we have

$$\text{dist}(\tau(x), \tau(z)) \le k^s \text{dist}(x,z)^\beta,$$

$$\text{dist}(\tau(z), \tau(H_{x,y}(z))) \le k^u \text{dist}(z, H_{x,y}(z))^\beta,$$

$$\text{dist}(\tau(H_{x,y}(z)), \tau(y)) \le k^s \text{dist}(H_{x,y}(z), y)^\beta.$$

Since the points x, y, and z are close, it is clear from the local product structure that

$$\text{dist}(x,z)^\beta + \text{dist}(z, H_{x,y}(z))^\beta + \text{dist}(H_{x,y}(z), y)^\beta \le k_5 \text{dist}(x,y)^\beta.$$

Hence, we obtain $\text{dist}(\tau(x), \tau(y)) \le k_6 \text{dist}(x,y)^\beta$ for all x and y in a set of full measure $\tilde{G} \subset G$. We can assume that $\tilde{G}$ is invariant by considering $\bigcap_{n=-\infty}^\infty f^n(\tilde{G})$. Since μ has full support, the set $\tilde{G}$ is dense in $\mathcal{M}$. Hence we can extend τ from $\tilde{G}$ and obtain an invariant Hölder-continuous conformal structure τ on $\mathcal{M}$. $\square$

4.4. **Proof of Proposition 2.4.** Let τ_0 be a continuous conformal structure on $\mathcal{E}$. We denote by $\tau_0(x)$ the conformal structure on $\mathcal{E}_x$, $x \in \mathcal{M}$. We consider the set

$$S(x) := \big\{(F_x^n)^* \big(\tau_0(f^n x)\big) : n \in \mathbb{Z}\big\}$$

in $\mathcal{C}(x)$, the space of conformal structures on $\mathcal{E}_x$. Here, F^* is the pullback action given by (3.6). Since F is uniformly quasiconformal, the sets $S(x)$ have uniformly bounded diameters. Since the space $\mathcal{C}(x)$ has nonpositive curvature, for every x there exists a uniquely determined ball of the smallest radius containing $S(x)$. We denote its center by $\tau(x)$.

It follows from the construction that the conformal structure τ is F-invariant and its distance from τ_0 is bounded. We also note that for any $k \geq 0$, the set $S_k(x) := \{(F_x^n)^*(\tau_0(f^n x)) : |n| \leq k\}$ depends continuously on x in Hausdorff distance, and so does the center $\tau_k(x)$ of the smallest ball containing $S_k(x)$. Since $S_k(x) \to S(x)$ as $k \to \infty$ for any x, the conformal structure τ is the pointwise limit of continuous conformal structures $\tau_k(x)$. Hence, τ is Borel measurable. $\qquad\square$

4.5. **Proof of Proposition 2.6.** We consider a fiber bundle $\mathcal{G}$ over $\mathcal{M}$ whose fiber over x is the Grassman manifold $\mathcal{G}_x$ of all k-dimensional subspaces in $\mathcal{E}_x$. The map $F_x : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ induces a natural map $\tilde{F}_x : \mathcal{G}_x \to \mathcal{G}_{f(x)}$. Thus we obtain a cocycle $\tilde{F} : \mathcal{G} \to \mathcal{G}$ over $f : \mathcal{M} \to \mathcal{M}$ given by $\tilde{F}(x, \xi) = (f(x), \tilde{F}_x(\xi))$ where $\xi \in \mathcal{G}_x$. Since the linear cocycle F and the bundle $\mathcal{E}$ are Hölder-continuous, both $\tilde{F}$ and $\tilde{F}^{-1}$ are Hölder-continuous and $\text{dist}_{C^1}(\tilde{F}_x, \tilde{F}_y) \leq k \, \text{dist}(x, y)^\beta$ for all $x, y \in \mathcal{M}$. Such $\tilde{F}$ is said to be in $C^\beta(f, \mathcal{G})$.

LEMMA 4.6. *There exists $C > 0$ such that for any $x \in \mathcal{M}$, subspaces $\xi, \eta \in \mathcal{G}_x$, $n \in \mathbb{Z}$, and $\epsilon > 0$, we have*

$$\text{dist}\big(\tilde{F}_x^n(\xi), \tilde{F}_x^n(\eta)\big) \leq C \cdot K_F(x, n) \cdot \text{dist}(\xi, \eta) \leq C \cdot C_\epsilon e^{\epsilon |n|} \cdot \text{dist}(\xi, \eta).$$

Proof. Let w and v be two unit vectors in $\mathcal{E}_x$. We denote $D = F_x^n$. Using the formula $2\langle Dw, Dv \rangle = \|Dw\|^2 + \|Dv\|^2 - \|Dw - Dv\|^2$ for the inner product, we obtain

$$\left[2\sin\left(\frac{\angle(Dw, Dv)}{2} \right) \right]^2 = 2\big(1 - \cos\angle(Dw, Dv)\big)$$

$$= 2 - \frac{2\langle Dw, Dv \rangle}{\|Dw\| \cdot \|Dv\|}$$

$$= \frac{2\|Dw\| \cdot \|Dv\| - \|Dw\|^2 - \|Dv\|^2 + \|Dw - Dv\|^2}{\|Dw\| \cdot \|Dv\|}$$

$$= \frac{\|Dw - Dv\|^2 - (\|Dw\| - \|Dv\|)^2}{\|Dw\| \cdot \|Dv\|}$$

$$\leq \frac{\|D\|^2 \cdot \|w - v\|^2}{\|Dw\| \cdot \|Dv\|}$$

$$\leq K_F(x, n)^2 \cdot \|w - v\|^2.$$

Suppose that the angle between the unit vectors w and v is sufficiently small so that it remains small when multiplied by $K_F(x, n)$. Then we obtain that the angle $\angle(Dw, Dv)$ is also small and

$$\angle(Dw, Dv) \leq C_0 \cdot K_F(x, n) \cdot \angle(w, v)$$

If the right-hand side is large the estimate is trivial, thus for any subspaces $\xi, \eta \in \mathcal{G}_x$ we have

$$\text{dist}\big(\tilde{F}_x^n(\xi), \tilde{F}_x^n(\eta)\big) \leq C \cdot K_F(x, n) \cdot \text{dist}(\xi, \eta),$$

where the distance between two subspaces is the maximal angle. We note that the maximal angle distance is Lipschitz equivalent to any smooth Riemannian metric on the Grassman manifold, and thus we have the estimate in any smooth metric on $\mathcal{G}$.

The case of $n < 0$ can be considered similarly. $\qquad\square$

The lemma implies that the expansion/contraction in the fiber is arbitrarily slow and, in particular, slower than the expansion/contraction of the hyperbolic system in the base. Hence the cocycle $\tilde{F}$ is *dominated* in the sense of [1, Definition 4.1]. This notion is similar to the notion of domination or bunching for partially hyperbolic systems, the difference is that in our context $\tilde{F}$ is not a diffeomorphism, only the maps $\tilde{F}_x$ are.

Dominated cocycles have Hölder-continuous strong (un)stable foliations. These subfoliate the weak (un)stable leaves, which are preimages of (un)stable leaves in the base. The strong stable foliation gives rise to an *s-holonomy* for $\tilde{F}$, an invariant family of maps between the fibers over the same stable leaf in the base. These facts are conveniently summarized in the following proposition, whose Lipschitz version appeared in [2, Proposition 4.1]. We will refer only to part (3), which in our setting can be easily obtained from Proposition 2.2 and its proof. Indeed, the desired holonomy $H^s_{x,y}$ is induced on the Grassmannians by $\lim_{n\to\infty}(F^n_x)^{-1} \circ F^n_y$.

PROPOSITION 4.7 ([1, Proposition 4.2]). *If the cocycle $\tilde{F} \in C^\beta(f, \mathcal{G})$ is dominated then there exists a unique partition $\mathcal{W}^s = \{\mathcal{W}^s(x,\xi) : (x,\xi) \in \mathcal{G}\}$ of the fiber bundle $\mathcal{G}$ such that*

1. *every $\mathcal{W}^s(x,\xi)$ is a β-Hölder graph over $W^s(x)$, with Hölder constant uniform on x;*
2. *$\tilde{F}(\mathcal{W}^s(x,\xi)) \subset \mathcal{W}^s(\tilde{F}(x,\xi))$ for all $(x,\xi) \in \mathcal{G}$;*
3. *the family of maps $H^s_{x,y} \colon \mathcal{G}_x \to \mathcal{G}_y$ defined for $y \in W^s(x)$ by $(y, H^s_{x,y}(\xi)) \in \mathcal{W}^s(x,\xi)$ is an s-holonomy for $\tilde{F}$.*

Moreover, there is a dual statement for strong unstable leaves.

Let μ be a measure on $\mathcal{M}$ with local product structure and full support. A μ-measurable F-invariant subbundle in $\mathcal{E}$ gives rise to a measurable $\tilde{F}$-invariant section $\phi \colon \mathcal{M} \to \mathcal{G}$. We denote by m the lift of μ to the graph Φ of ϕ, i.e., for a set $X \subset \mathcal{G}$, $m(X) = \mu(\pi(X \cap \Phi))$, where $\pi \colon \mathcal{G} \to \mathcal{M}$ is the projection. Alternatively, m can be defined by specifying that for μ-almost every x in $\mathcal{M}$ the conditional measure m_x in the fiber $\mathcal{G}_x$ is the atomic measure at $\phi(x)$. Since μ is f-invariant and Φ is $\tilde{F}$-invariant, the measure m is $\tilde{F}$-invariant.

Lemma 4.6 implies that Lyapunov exponent of $\tilde{F}$ along the fiber is zero at every $\xi \in \mathcal{G}$, in particular the exponent of m along the fiber is zero. This together with the existence of s- and u-holonomies for $\tilde{F}$ allows us to apply [2, Theorem C] to the measure m and conclude that there exists a system of conditional measures $\tilde{m}_x$ on $\mathcal{G}_x$ for m which are holonomy invariant and vary continuously on x in $\operatorname{supp}\mu = \mathcal{M}$. Since the systems of conditional measures $\{m_x\}$

and $\{\tilde{m}_x\}$ coincide on a set X of full μ measure, we see that $\tilde{m}_x = m_x$ is the atomic measure at $\phi(x)$ for all $x \in X$. Since X is dense we obtain that $\tilde{m}_x$ is atomic for all $x \in \mathcal{M}$. Indeed, by compactness of $\mathcal{G}$, for any $x \in \mathcal{M}$ we can take a sequence $X \ni x_i \to x$ such that $\phi(x_i)$ converge to some $\xi \in \mathcal{G}_x$. This implies that $\tilde{m}_{x_i} = m_{x_i}$ converge to the atomic measure at ξ, which therefore coincides with $\tilde{m}_x$ by continuity of the family $\{\tilde{m}_x\}$. Denoting $\tilde{\phi}(x) = \operatorname{supp} \tilde{m}_x$ for $x \in \mathcal{M}$, we obtain a continuous section $\tilde{\phi}$ which coincides with ϕ on X. Since $\tilde{\phi}$ is invariant under s- and u-holonomies we conclude that it is β-Hölder. This yields that the invariant measurable subbundle in $\mathcal{E}$ coincides μ-almost everywhere with a Hölder-continuous one. $\qquad \square$

4.6. **Proof of Proposition 2.7.** We use the following particular case of Zimmer's Amenable Reduction Theorem, see [4, Corollary 1.8] and [3, Theorem 3.5.9].

ZIMMER'S AMENABLE REDUCTION THEOREM. *Let f be an ergodic transformation of a measure space (X, μ) and let $F: X \to GL(d, \mathbb{R})$ be a measurable function. Then, there exists a measurable function $C: X \to GL(d, \mathbb{R})$ such that the function $G(x) = C^{-1}(f(x))F(x)C(x)$ takes values in a maximal amenable subgroup of $GL(d, \mathbb{R})$.*

It is known that any maximal amenable subgroup of $GL(d, \mathbb{R})$ is conjugate to a group of block-triangular matrices of the form

$$\begin{pmatrix} A_1 & * & \dots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_r \end{pmatrix}$$

where each diagonal block A_i is a scalar multiple of a $d_i \times d_i$ orthogonal matrix and $d_1 + \dots + d_r = d$.

COROLLARY 4.8. *Let f be a diffeomorphism of a smooth compact manifold $\mathcal{M}$ preserving an ergodic measure μ and let $F: \mathcal{E} \to \mathcal{E}$ be a measurable linear cocycle over f. Then, F preserves a measurable conformal structure either on $\mathcal{E}$ or on a measurable invariant proper nontrivial subbundle of $\mathcal{E}$.*

Proof. We recall that $\mathcal{E}$ can be trivialized on a set of full μ-measure [3, Proposition 2.1.2], so we measurably identify $\mathcal{E}$ with $\mathcal{M} \times \mathbb{R}^d$ and view F as a function $\mathcal{M} \to GL(d, \mathbb{R})$. Thus can we apply the Amenable Reduction Theorem to F and obtain a measurable coordinate change function $C: \mathcal{M} \to GL(d, \mathbb{R})$ such that $G_x = C^{-1}(f(x))F_x C(x)$ is of the above form for μ-almost all $x \in \mathcal{M}$. If $r = 1$ and $d_1 = d$, we obtain that G_x is a scalar multiple of a $d \times d$ orthogonal matrix. This implies that F is conformal with respect to the pull back by C^{-1} of the standard conformal structure on $\mathbb{R}^d$. This gives a measurable invariant conformal structure for F on $\mathcal{E}$. If $r > 1$, then the last block A_r acts conformally on a d_r-dimensional invariant subspace for G. Again pulling back by C^{-1} we obtain a measurable invariant subbundle with conformal structure for F. $\qquad \square$

Now suppose that that the system satisfies the assumptions of Proposition 2.7. We apply Corollary 4.8 with μ being the measure of maximal entropy for f. If we have a measurable invariant conformal structure on $\mathscr{E}$ then it is Hölder-continuous by Proposition 2.3. If we have a measurable invariant subbundle $\mathscr{E}'$ then it is Hölder-continuous by Proposition 2.6. The restriction of F to $\mathscr{E}'$ is a Hölder-continuous cocycle which also satisfies the same pinching assumption as F. Hence the invariant measurable conformal structure on $\mathscr{E}'$ is again Hölder-continuous by Proposition 2.3. □

4.7. **Proof of Theorem 1.1.** By Proposition 2.1, the assumption of the theorem implies that (2.1) is satisfied with $\gamma = 0$. Hence the conclusions of Proposition 2.2 hold, which has the following implication for quasiconformal distortion.

LEMMA 4.9. *Let A and B be two linear transformations of $\mathbb{R}^d$. Suppose that either $\|A^{-1}B - \mathrm{Id}\| \le r$ or $\|AB^{-1} - \mathrm{Id}\| \le r$, where $r < 1$. Then,*
$$\frac{1-r}{1+r} \le \frac{K(A)}{K(B)} \le \frac{1+r}{1-r},$$
where $K(A)$ and $K(B)$ are quasiconformal distortions of A and B, respectively.

Proof. We recall that $K(A) = K(A^{-1})$ and $K(A_1 A_2) \le K(A_1) K(A_2)$. Suppose that $\|A^{-1}B - \mathrm{Id}\| \le r$. Denoting $A^{-1}B - \mathrm{Id} = R$ and multiplying by A, we have $B = A(\mathrm{Id} + R)$. Since for any unit vector v, $1 - r \le \|(\mathrm{Id} + R)v\| \le 1 + r$, we obtain
$$K(B) \le K(A) K(\mathrm{Id} + R) \le K(A) \cdot (1 + r)/(1 - r).$$
Multiplying by B^{-1}, we have $A^{-1} = (\mathrm{Id} + R)B^{-1}$ and hence
$$K(A) = K(A^{-1}) \le K(\mathrm{Id} + R) K(B^{-1}) \le (1 + r)/(1 - r) \cdot K(B).$$
The case of $\|AB^{-1} - \mathrm{Id}\| \le r$ is similar. □

Now we will show that F is uniformly quasiconformal using a dense orbit argument. Since f is transitive, there exists a point $z \in \mathcal{M}$ with dense orbit $\mathcal{O} = \{f^k z : k \in \mathbb{Z}\}$. We will show that the quasiconformal distortion $K_F(z, k)$ is uniformly bounded in $k \in \mathbb{Z}$. Since $\mathcal{O}$ is dense and $K_F(x, k)$ is continuous on $\mathcal{M}$ for each k, this implies that $K_F(x, k)$ is uniformly bounded in $x \in X$ and $k \in \mathbb{Z}$.

We consider any two points of $\mathcal{O}$ with $\mathrm{dist}(f^{k_1} z, f^{k_2} z) < \delta_0$, where δ_0 is sufficiently small to apply the Anosov Closing Lemma [10, Theorem 6.4.15]. We assume that $k_1 < k_2$ and denote $x = f^{k_1} z$ and $n = k_2 - k_1$, so $\delta = \mathrm{dist}(x, f^n x) < \delta_0$. By the Anosov Closing Lemma, there exists $p \in X$ with $f^n p = p$ such that $\mathrm{dist}(f^i x, f^i p) \le c\delta$ for $i = 0, \ldots, n$.

Let y be a point in $W^s_{\mathrm{loc}}(p) \cap W^u_{\mathrm{loc}}(x)$. Then by Proposition 2.2,
$$\left\| (F^n_p)^{-1} \circ F^n_y - \mathrm{Id} \right\| \le C\delta^\beta \quad \text{and} \quad \left\| (F^{-n}_y)^{-1} \circ F^{-n}_x - \mathrm{Id} \right\| \le C\delta^\beta.$$
Hence by Lemma 4.9,
$$\frac{K_F(y, n)}{K_F(p, n)} \le \frac{1 + C\delta^\beta}{1 - C\delta^\beta} \le 2 \quad \text{and} \quad \frac{K_F(x, n)}{K_F(y, n)} \le 2,$$
if δ_0 is sufficiently small. Thus, $K_F(x, n) \le 4K_F(p, n) \le 4C_{\mathrm{per}}$.

We take $m > 0$ such that the set $\{f^j(z) : |j| \le m\}$ is δ_0-dense in $\mathcal{M}$. Let $K_m = \max\{K_F(z, j) : |j| \le m\}$. Then for any $k > m$, there exists j with $|j| \le m$ such that $\text{dist}(f^k(z), f^j(z)) \le \delta_0$. Hence,

$$K_F(z, k) \le K_F(z, j) \cdot K_F(f^j(z), k - j) \le K_m \cdot 4C_{\text{per}}.$$

The case of $k < -m$ is considered similarly. Thus, $K_F(z, k)$ is uniformly bounded and hence so is $K_F(x, k)$ for all $x \in \mathcal{M}$ and $k \in \mathbb{Z}$.

Thus, F is uniformly quasiconformal on $\mathcal{E}$. It follows from Corollary 2.5 that there exists a Hölder-continuous metric on $\mathcal{E}$ with respect to which F is conformal.

Now we prove the second part of the theorem. We observe that $K_F(p, n) = \|F_p^n\| \cdot \|(F_p^n)^{-1}\| \le (C_{\text{per}}')^2$ and hence F is conformal with respect to a Hölder-continuous Riemannian metric g on $\mathcal{E}$. This means that there exists a positive Hölder-continuous function $a(x)$ such that for each $x \in \mathcal{M}$ and each $v \in \mathcal{E}_x$

$$\|F_x(v)\|_{g(f(x))} = a(x) \cdot \|v\|_{g(x)}.$$

It remains to renormalize the metric g. For a positive function $\varphi(x)$, we consider a new metric $\tilde{g}(x) = g(x)/\varphi(x)$. Then we have

$$\varphi(f(x)) \cdot \|F_x(v)\|_{\tilde{g}(f(x))} = a(x)\varphi(x) \cdot \|v\|_{\tilde{g}(x)}.$$

Thus, we need to find a Hölder-continuous function $\varphi(x)$ such that $\varphi(f(x)) = a(x)\varphi(x)$, i.e.,

$$(4.7) \qquad\qquad a(x) = \varphi(f(x))/\varphi(x) \qquad \text{for all } x \in \mathcal{M}.$$

Let p be a periodic point of period n. Then, $a(p)a(fp)\cdots a(f^{n-1}p) = \|F_p^n\|_{g(p)}$. If $\|F_p^n\|_{g(p)} > 1$, then

$$\|F^{mn}\|_{g(p)} = \|F_p^n\|_{g(p)}^m \to \infty \qquad \text{as } m \to \infty.$$

If $\|F_p^n\|_{g(p)} < 1$, then

$$\left\|(F^{mn})^{-1}\right\|_{g(p)} = \left\|(F_p^n)^{-1}\right\|_{g(p)}^m = \left\|F_p^n\right\|_{g(p)}^{-m} \to \infty \qquad \text{as } m \to \infty.$$

Either case contradicts the assumption of Theorem 1.1. Therefore, we must have $a(p)a(fp)\cdots a(f^{n-1}p) = 1$ for any periodic point p. Now, the Livšic Theorem [10, Theorem 19.2.1] implies that equation (4.7) has a Hölder-continuous solution $\varphi(x)$, and F is an isometry with respect to the Hölder-continuous metric $\tilde{g}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.8. **Proof of Theorem 1.3.** By Proposition 2.1, the assumption on the periodic data implies that (2.1) is satisfied with $\gamma = 0$. Now Proposition 2.7 yields that F preserves either a Hölder-continuous one-dimensional subbundle in $\mathcal{E}$ or a Hölder-continuous invariant conformal structure on $\mathcal{E}$.

Suppose that there is a continuous invariant one-dimensional subbundle. Hence for any point p and any n with $f^n p = p$ we obtain an invariant line for $F_p^n : \mathcal{E}_p \to \mathcal{E}_p$. This implies that the eigenvalues of F_p^n are real. Hence, by the assumption of the theorem, they are either λ, λ or $\lambda, -\lambda$. If F is orientation-preserving, the former is always the case. It follows that $F_p^n = \lambda \cdot \text{Id}$ since it

is diagonalizable, and hence $K_F(p, n) = 1$. For such F we can apply Theorem 1.1 and obtain a Hölder-continuous metric on $\mathcal{E}$ with respect to which F is conformal.

If $\mathcal{E}$ is not orientable, we can pass to a double cover. If F is orientation-reversing, we can consider cocycle F^2 over f^2. Thus we can always obtain an orientation-preserving cocycle F' which by the above is conformal. This implies uniform quasiconformality of the original cocycle F. Now Corollary 2.5 yields conformality of F.

We conclude that F is conformal with respect to a Hölder-continuous metric on $\mathcal{E}$. The second part can be establishes in the same way as in the proof Theorem 1.1. Indeed, the assumption implies that for any periodic point p the map F_p^n is conjugate to an orthogonal matrix and hence there exists a constant $C(p)$ such that $\max\{\|F_p^n\|, \|(F_p^n)^{-1}\|\} \le C(p)$ for any period n of p. $\qquad\square$

4.9. **Proof of Proposition 1.2.** The idea of the example was suggested to us by M. Guysinsky and is similar to his example in [5]. We recall that since the tangent bundle is trivial, a cocycle $F: \mathcal{E} \to \mathcal{E}$ is defined by a function $A: \mathcal{M} \to GL(d, \mathbb{R})$ via $F(x, v) = (f(x), A(x)v)$. First we construct an example for $d = 3$.

Let S be a closed, and hence compact, f-invariant set in $\mathcal{M}$ that does not contain any periodic points (such sets always exist for Anosov systems and can be constructed using symbolic dynamics). Let $\alpha: \mathcal{M} \to \mathbb{R}$ be given by

$$\alpha(x) = \operatorname{dist}(x, S) \cdot \epsilon / (2 \operatorname{diam} \mathcal{M}).$$

Then α is Lipschitz, $\alpha(x) = 0$ for all $x \in S$, and $0 < \alpha(x) \le \epsilon/2$ for all $x \notin S$. We set

$$A(x) = \begin{pmatrix} \cos \tilde{\alpha}(x) & -\sin \tilde{\alpha}(x) & \epsilon \\ \sin \tilde{\alpha}(x) & \cos \tilde{\alpha}(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $\tilde{\alpha}$ is a modification of the function α constructed below. For a point $p \in \mathcal{M}$ and $n \in \mathbb{N}$, we denote

$$A(p, n) = A(f^{n-1}p) \cdots A(fp) \cdot A(p),$$
$$\tilde{\alpha}(p, n) = \tilde{\alpha}(f^{n-1}p) + \cdots + \tilde{\alpha}(fp) + \tilde{\alpha}(p).$$

Then we have

$$A(p, n) = \begin{pmatrix} \cos \tilde{\alpha}(p, n) & -\sin \tilde{\alpha}(p, n) & * \\ \sin \tilde{\alpha}(p, n) & \cos \tilde{\alpha}(p, n) & * \\ 0 & 0 & 1 \end{pmatrix}$$

Let p be a periodic point and let n be its minimal period. Since the eigenvalues of $A(p, n)$ are of modulus 1, it is conjugate to an orthogonal matrix if and only if it is diagonalizable over $\mathbb{C}$. For $A(p, n)$ to be diagonalizable, it suffices to have three different eigenvalues, which is equivalent to $\tilde{\alpha}(p, n) \ne \pi k$. The function α does not necessarily satisfy this condition at every periodic point, so we modify it inductively to obtain a function $\tilde{\alpha}$ that does.

Since there are countably many periodic orbits for f, we can order them $\{\mathcal{O}_1, \mathcal{O}_2, \ldots\}$. Let $\alpha_0 = \alpha$ and suppose that α_{m-1} is defined. If $\alpha_{m-1}(p, n_m) \ne$

πk, where $p \in \mathcal{O}_m$ and n_m is the minimal period of p, we set $\alpha_m = \alpha_{m-1}$. If $\alpha_{m-1}(p, n_m) = \pi k$, we modify α_{m-1} in a small neighborhood of $\mathcal{O}_m$ to obtain α_m. Let

$$\delta_m = (1/2) \operatorname{dist}\big(\mathcal{O}_m, (\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{m-1} \cup S)\big).$$

We change α_{m-1} in the δ_m-neighborhood of $\mathcal{O}_m$ so that the new function α_m is Lipschitz on $\mathcal{M}$, $\alpha_m(p, n_m) \neq \pi k$, and the Lipschitz norm of $\alpha_{m-1} - \alpha_m$ is less than $\epsilon/(4m^2)$. Clearly $\alpha_m = \alpha_{m-1}$ on S and on $\mathcal{O}_1, \ldots, \mathcal{O}_{m-1}$. We define $\tilde{\alpha}(x) = \lim_{m\to\infty} \alpha_m(x)$. It follows from the construction that $\tilde{\alpha}(x)$ is Lipschitz, $\tilde{\alpha}(x) = 0$ on S, and $\tilde{\alpha}(p, n) \neq \pi k$ for any periodic point p of a minimal period n. Thus, the matrix $A(p, n)$ has three different eigenvalues: $e^{i\tilde{\alpha}(p,n)}$, $e^{-i\tilde{\alpha}(p,n)}$, 1. Hence, $A(p, n)$ is diagonalizable. We also note that since $\max|\alpha(x) - \tilde{\alpha}(x)| \leq \epsilon/2$, we have $\max|\tilde{\alpha}(x)| \leq \epsilon$, and thus $A(x)$ is ϵ-close to the identity.

Now we show that the cocycle F is not uniformly quasiconformal. Let x be a point in S. Then x is nonperiodic and $\tilde{\alpha}(f^n x) = 0$ for every n. Hence,

$$A(x) = \begin{pmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A(x, n) = \begin{pmatrix} 1 & 0 & n\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The quasiconformal distortion of F is not uniformly bounded along the orbit of x, since for the first coordinate vector $\|F_x^n v_1\| = \|A(x, n) v_1\| \to \infty$, while for the second coordinate vector $\|F_x^n v_2\| = \|A(x, n) v_2\| = 1$. It follows that F cannot be conformal with respect to any continuous Riemannian metric on $\mathcal{E}$.

This example can be extended to any dimension $d \geq 4$ by considering

$$F(x, v) = (f(x), \tilde{A}(x) v) \qquad \text{with} \qquad \tilde{A}(x) = \begin{pmatrix} A(x) & 0 \\ 0 & \operatorname{Id}_{d-3} \end{pmatrix}$$

where Id_{d-3} is the $(d-3) \times (d-3)$ identity matrix. $\square$

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BORIS KALININ <kalinin@jaguar1.usouthal.edu>: Department of Mathematics & Statistics, University of South Alabama, Mobile, AL 36688, USA

VICTORIA SADOVSKAYA <sadovska@jaguar1.usouthal.edu>: Department of Mathematics & Statistics, University of South Alabama, Mobile, AL 36688, USA