

NON-STATIONARY NORMAL FORMS FOR CONTRACTING EXTENSIONS

BORIS KALININ*

Dedicated to the memory of Anatole Katok

ABSTRACT. We present the theory of non-stationary normal forms for uniformly contracting smooth extensions with sufficiently narrow Mather spectrum. We give coherent proofs of existence, (non)uniqueness, and a description of the centralizer results. As a corollary, we obtain corresponding results for normal forms along an invariant contracting foliation. The main improvements over the previous results in the narrow spectrum setting include explicit description of non-uniqueness and obtaining results in any regularity above the precise critical level, which is especially useful for the centralizer. In addition to sub-resonance normal form, we also prove corresponding results for resonance normal form, which is new in the narrow spectrum setting.

1. INTRODUCTION

The theory of normal forms for smooth maps goes back to Poincaré and Sternberg [St57] and plays an important role in dynamics. It has been extensively studied in the classical setting of normal forms at fixed points and invariant manifolds [BrKo]. The theory of non-stationary normal forms was developed more recently in the context of extensions and invariant foliations. The primary motivations and applications were various rigidity results for systems and actions exhibiting some form of hyperbolicity.

In the setting of an invariant contracting foliation \mathcal{W} for a diffeomorphism f of a compact manifold X , the goal is to obtain a family of diffeomorphisms $\mathcal{H}_x : \mathcal{W}_x \rightarrow T_x\mathcal{W}$ such that the maps

$$(1.1) \quad \mathcal{P}_x = \mathcal{H}_{fx} \circ f \circ \mathcal{H}_x^{-1} : T_x\mathcal{W} \rightarrow T_{fx}\mathcal{W}$$

are as simple as possible. The maps \mathcal{P}_x , called the normal form of f on \mathcal{W} , will be polynomial in our setting. The case of linear \mathcal{P}_x is called non-stationary linearization. Some of the theory can be developed in a more general context of a smooth extension \mathcal{F} of f to a vector bundle \mathcal{E} over X . The foliation setting produces such a smooth extension on the tangent bundle to the foliation $\mathcal{E} = T\mathcal{W}$ as follows. We take $\mathcal{F}_x =$

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$f|_{\mathcal{W}_x} : \mathcal{E}_x \rightarrow \mathcal{E}_{f_x}$ after local identification of the leaf \mathcal{W}_x with its tangent space $\mathcal{E}_x = T_x\mathcal{W}$. In this setting, the map \mathcal{H}_x is a coordinate change on \mathcal{E}_x .

The non-stationary linearization for one-dimensional extensions was obtained by Katok and Lewis [KtL91] and applied to the study of rigidity for $SL(n, \mathbb{Z})$ actions on \mathbb{T}^n . For higher-dimensional foliations under the assumption of constant $1/2$ pinched bounds for contraction rates, non-stationary linearization follows from results of Guysinsky and Katok [GuKt98] or from results of Feres in [Fe95]. Under a weaker assumption of point-wise $1/2$ pinching, it was obtained by Sadovskaya [S05] and some further properties were established by Kalinin and Sadovskaya [KS06]. This is a less technical but very important case of non-stationary normal forms and we give a brief summary of these results in Section 2. They were used extensively in the study of rigidity of Anosov systems and higher rank actions, see e.g. [S05, KS03, KS06, F07, FFH10, GKS11, Bu18].

In higher-dimensional setting without $1/2$ pinching, there may be no smooth non-stationary linearization, and so a polynomial normal form is sought. Under the narrow band spectrum assumption, such forms were developed by Guysinsky and Katok [GuKt98, Gu02] and used by Katok and Spatzier to obtain local rigidity of algebraic higher rank Anosov abelian actions [KtSp97]. A geometric point of view on normal forms was developed by Feres in [Fe04]. The narrow band assumption ensures that the polynomial maps involved belong to a finite dimensional Lie group of so called sub-resonance generated polynomials. In [KS16] Kalinin and Sadovskaya obtained stronger results, constructing \mathcal{H}_x which depend smoothly on x along the leaves and proving that they define an atlas with transition maps in a similar finite dimensional Lie group.

A parallel theory of non-stationary normal forms was also developed for non-uniform contractions. Basic results were formulated by Kalinin and Katok [KKt01] along with a program of applications to measure rigidity for non-uniformly hyperbolic systems and actions. The existence of \mathcal{H}_x for a general contracting C^∞ extension was proved by Li and Lu [LL05] in the setting of random dynamical systems. Some results, such as existence of Taylor polynomial or formal series for \mathcal{H}_x , can be obtained for extensions more general than contractions, see [AK92, A, LL05]. Non-stationary linearization of a $C^{1+\alpha}$ diffeomorphism along a non-uniformly contracting foliation was obtained by Kalinin and Katok [KKt07] for one-dimensional leaves and by Katok and Rodriguez Hertz [KtR15] under pinching assumption on Lyapunov exponents. The former result was used in the study of measure rigidity by Kalinin, Katok, and Rodriguez Hertz [KKt07, KKtR11]. Following [KS16], the advanced theory of normal forms on non-uniformly contracting foliations, including the consistency of normal form coordinates along the leaves, were obtained independently by Kalinin and Sadovskaya [KS17] and, in differential geometric formulations, by Melnick [M19]. In addition to measurable properties of non-uniformly hyperbolic systems, this theory is useful in global, rather than local, smooth rigidity of uniformly hyperbolic systems, where the spectrum may

not be narrow [FKSp11]. The main disadvantage of this setting is that dependence of \mathcal{H}_x on x is only measurable.

Our main goal is to present the theory of non-stationary normal forms, both in extension and foliation settings, assuming that the spectrum is sufficiently narrow. In particular, the results apply to perturbations of algebraic and point spectrum systems. We follow the approach developed in [KS16, KS17] and give a coherent treatment of existence, (non)uniqueness, and centralizer results. The main improvements over the previous results in the narrow band spectrum setting are the following. Our construction allows us to describe the exact extent of non-uniqueness in \mathcal{H}_x and \mathcal{P}_x and hence gives the description of the centralizer. It works in any regularity of \mathcal{F} above the precise critical level in Hölder classes. This is especially important for the centralizer results as they yield an automatic bootstrap of regularity for a commuting system, from critical one to that of \mathcal{H}_x . These improvements proved useful in rigidity results [DWX19, GKS19]. In addition to sub-resonance normal form, we also prove existence, (non)uniqueness, and centralizer results for resonance normal form, which has not been done in the narrow band setting.

2. NON-STATIONARY LINEARIZATION

In this section we summarize the results on non-stationary linearization, that is existence of \mathcal{H}_x so that \mathcal{P}_x in (1.1) are linear. This important particular case of non-stationary normal form theory is easier to formulate and provides a point of comparison for the more technical general case. Also, it is the only result obtained under pointwise assumption on contraction rates. We formulate it in foliation setting, where it is most interesting. We state the results for C^∞ case, as established in [S05, Proposition 4.1, Lemma 4.1] and [KS06, Proposition 4.6], and then make remarks on lower regularity.

Theorem 2.1 (Non-stationary linearization). [S05, KS06]

Let f be a diffeomorphism of a compact manifold X and let \mathcal{W} be an f -invariant continuous foliation of X with uniformly C^∞ leaves. Suppose that $\|Df|_{T\mathcal{W}}\| < 1$, and there exist $C > 0$ and $\gamma < 1$ such that

$$(2.1) \quad \|(Df^n|_{T_x\mathcal{W}})^{-1}\| \cdot \|Df^n|_{T_x\mathcal{W}}\|^2 \leq C\gamma^n \quad \text{for all } x \in X, n \in \mathbb{N}.$$

Then for every $x \in X$ there exists a C^∞ diffeomorphism $\mathcal{H}_x : \mathcal{W}_x \rightarrow T_x\mathcal{W}$ such that

- (i) $\mathcal{H}_{fx} \circ f \circ \mathcal{H}_x^{-1} = Df|_{T_x\mathcal{W}}$,
- (ii) $\mathcal{H}_x(x) = 0$ and $D_x\mathcal{H}_x$ is the identity map,
- (iii) \mathcal{H}_x depends continuously on $x \in X$ in C^∞ topology.
- (iv) Such a family \mathcal{H}_x is unique and depends smoothly on x along the leaves of \mathcal{W} .
- (v) The map $\mathcal{H}_y \circ \mathcal{H}_x^{-1} : T_x\mathcal{W} \rightarrow T_y\mathcal{W}$ is affine for any $x \in X$ and $y \in \mathcal{W}_x$. Hence the non-stationary linearization \mathcal{H} defines affine structures on the leaves of \mathcal{W} .

We note that non-stationary linearization is the only case when the \mathcal{H}_x , and hence \mathcal{P}_x , are unique under assumptions (i) and (ii) and sufficient regularity. This uniqueness immediately implies the description of centralizer as in part (3) of Theorem 4.6. Together with (v), it also easily gives smooth dependence on x along the leaves of \mathcal{W} .

Under stronger $1/2$ pinching assumption on rates in place of (2.1), finite regularity version follows from our general results, see Corollary 4.8. However, finite regularity results can also be obtained using (2.1):

Remark 2.2. *The proof of existence of \mathcal{H} in [S05, Proposition 4.1] is for C^N with N sufficiently large and can be seen to work for $N = 2$.*

Parts (iv) and (v) hold under the assumption that f and \mathcal{H}_x are C^2 . This is clear from the proof of [S05, Lemma 4.1] and [KS06, Proposition 4.6]. More precisely, it suffices to assume that $Df|_{T_x\mathcal{W}}$ and $D\mathcal{H}_x$ are Lipschitz along the leaves of \mathcal{W} with uniform constant for all local leaves. More generally, uniqueness holds if they are α -Hölder under stronger pinching assumption with the term $\|Df^n|_{T_x\mathcal{W}}\|^2$ in (2.1) replaced by $\|Df^n|_{T\mathcal{W}}\|^{1+\alpha}$ (cf. Corollary 4.8). In the particular case when \mathcal{W} is a one-dimensional foliation, the uniqueness holds if \mathcal{H} is C^1 [KtL91].

We note that equation (2.1) with $1 + \beta$ in place of 2 is precisely the *fiber bunching* assumption on the linear cocycle $Df|_{T_x\mathcal{W}}$ relative to the contraction along \mathcal{W} , which in particular ensures existence of cocycle holonomies along \mathcal{W} . In fact, the holonomies are given by the derivatives of the transition maps in (v).

3. PRELIMINARIES AND NOTATIONS

3.1. Smooth extensions. Let \mathcal{E} be a continuous vector bundle over a compact metric space X , let \mathcal{V} be a neighborhood of the zero section in \mathcal{E} , and let f be a homeomorphism of X . We consider an extension $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{E}$ that projects to f and preserves the zero section. We assume that the corresponding fiber maps $\mathcal{F}_x : \mathcal{V}_x \rightarrow \mathcal{E}_{f(x)}$ are C^r diffeomorphisms.

If $r = N \in \mathbb{N}$, we will assume that \mathcal{F}_x depend continuously on x in C^N topology. To obtain sharper results, we will also use *Hölder condition at the zero section*. We assume that the fibers \mathcal{E}_x are equipped with a continuous family of Riemannian norms. We denote by $B_{x,\sigma}$ the closed ball of radius $\sigma > 0$ centered at $0 \in \mathcal{E}_x$. For $N \in \mathbb{N}$ and $0 \leq \alpha \leq 1$ we denote by $C^{N,\alpha}(B_{x,\sigma}) = C^{N,\alpha}(B_{x,\sigma}, \mathcal{E}_{f_x})$ the space of functions $R : B_{x,\sigma} \rightarrow \mathcal{E}_{f_x}$ with continuous derivatives up to order N on $B_{x,\sigma}$ and, if $\alpha > 0$, with N^{th} derivative satisfying α -Hölder condition at 0:

$$(3.1) \quad \|D^{(N)}R\|_\alpha = \sup \{ \|D_t^{(N)}R - D_0^{(N)}R\| \cdot \|t\|^{-\alpha} : 0 \neq t \in B_{x,\sigma} \} < \infty.$$

We call $\|D^{(N)}R\|_\alpha$ the α -Hölder constant of $D^{(N)}R$ at 0. We equip the space $C^{N,\alpha}(B_{x,\sigma})$ with the norm

$$(3.2) \quad \|R\|_{C^{N,\alpha}(B_{x,\sigma})} = \max \{ \|R\|_0, \|D^{(1)}R\|_0, \dots, \|D^{(N)}R\|_0, \|D^{(N)}R\|_\alpha \},$$

where $\|D^{(k)}R\|_0 = \sup \{ \|D_t^{(k)}R\| : t \in B_{x,\sigma} \}$ and the last term is omitted if $\alpha = 0$.

Definition 3.1. We say that \mathcal{F} is a $C^{N,\alpha}$ extension of f , $N \in \mathbb{N}$ and $0 \leq \alpha \leq 1$, if for some $\sigma > 0$ the fiber maps $\mathcal{F}_x : B_{x,\sigma} \rightarrow \mathcal{E}_{f(x)}$ are $C^{N,\alpha}$ diffeomorphisms which depend continuously on x in C^N topology and the norms $\|\mathcal{F}_x\|_{C^{N,\alpha}(B_{x,\sigma})}$ are uniformly bounded.

Similarly, we say that $\mathcal{H} = \{\mathcal{H}_x\}_{x \in X}$, where $\mathcal{H}_x : B_{x,\sigma} \rightarrow \mathcal{E}_{f(x)}$, is a $C^{N,\alpha}$ coordinate change if it is a $C^{N,\alpha}$ extension of $f = \text{Id}$ which preserves the zero section.

3.2. Mather spectrum of the derivative. For a smooth extension \mathcal{F} we will denote by F its derivative of at the zero section, that is $F : \mathcal{E} \rightarrow \mathcal{E}$ is a continuous linear extension of f whose fiber maps are linear isomorphisms $F_x = D_0 \mathcal{F}_x : \mathcal{E}_x \rightarrow \mathcal{E}_{f(x)}$. Such a linear extension F induces a bounded linear operator F^* on the space of continuous sections of \mathcal{E} by $F^*v(x) = F(v(f^{-1}x))$. The spectrum $Sp F^*$ of complexification of F^* is called *Mather spectrum* of F . Under a mild assumption that non-periodic points of f are dense in X , the Mather spectrum consists of finitely many closed annuli centered at 0, see e.g. [P], and its *characteristic set* $\Lambda(F) = \{\lambda \in \mathbb{R} : \exp \lambda \in Sp F^*\}$ consists of finitely many closed intervals. We will assume that F is a contraction and that the Mather spectrum of F is sufficiently narrow.

Definition 3.2. Let $\varepsilon > 0$ and $\chi = (\chi_1, \dots, \chi_\ell)$, where $\chi_1 < \dots < \chi_\ell < 0$. We say that a linear extension F has (χ, ε) -spectrum if

$$(3.3) \quad \Lambda(F) = \{\lambda \in \mathbb{R} : \exp \lambda \in Sp F^*\} \subset \bigcup_{i=1}^{\ell} (\chi_i - \varepsilon, \chi_i + \varepsilon)$$

If F has (χ, ε) -spectrum with disjoint intervals then the bundle \mathcal{E} splits into direct sum

$$(3.4) \quad \mathcal{E} = \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^\ell$$

of continuous F -invariant sub-bundles so that $\Lambda(F|_{\mathcal{E}^i})$ is contained in $(\chi_i - \varepsilon, \chi_i + \varepsilon)$. This can be expressed using a convenient metric [GuKt98]: for each $i = 1, \dots, \ell$ there exists a continuous family of Riemannian norms $\|\cdot\|_x$ on \mathcal{E}_x such that the splitting (3.4) is orthogonal and

$$(3.5) \quad e^{\chi_i - \varepsilon} \|t\|_x \leq \|F_x(t)\|_{f(x)} \leq e^{\chi_i + \varepsilon} \|t\|_x \quad \text{for every } t \in \mathcal{E}_x^i.$$

We will equip \mathcal{E} with such a norm and will suppress the dependence on x . We can also summarize (3.5) using operator norms

$$(3.6) \quad \|F|_{\mathcal{E}_x^i}\| \leq e^{\chi_i + \varepsilon}, \quad \|(F|_{\mathcal{E}_x^i})^{-1}\| \leq e^{-\chi_i + \varepsilon}, \quad \|F_x\| \leq e^{\chi_\ell + \varepsilon}, \quad \|(F_x)^{-1}\| \leq e^{-\chi_1 + \varepsilon}.$$

3.3. Sub-resonance and resonance polynomials. We say that a map between vector spaces is *polynomial* if each component is given by a polynomial in some, and hence every, basis. We will consider a polynomial map $P : \mathcal{E}_x \rightarrow \mathcal{E}_y$ with $P(0_x) = 0_y$ and split it into components $(P_1(t), \dots, P_\ell(t))$, where $P_i : \mathcal{E}_x \rightarrow \mathcal{E}_y^i$. Each P_i can be written uniquely as a linear combination of polynomials of specific homogeneous types. We say that $Q : \mathcal{E}_x \rightarrow \mathcal{E}_y^i$ has *homogeneous type* $s = (s_1, \dots, s_\ell)$, where s_1, \dots, s_ℓ are non-negative integers, if for any real numbers a_1, \dots, a_ℓ and vectors $t_j \in \mathcal{E}_x^j$, $j = 1, \dots, \ell$,

we have

$$(3.7) \quad Q(a_1 t_1 + \cdots + a_\ell t_\ell) = a_1^{s_1} \cdots a_\ell^{s_\ell} \cdot Q(t_1 + \cdots + t_\ell).$$

Definition 3.3. We say that a homogeneous type $s = (s_1, \dots, s_\ell)$ for $P_i : \mathcal{E}_x \rightarrow \mathcal{E}_y^i$ is

$$(3.8) \quad \text{sub-resonance if } \chi_i \leq \sum_{j=1}^{\ell} s_j \chi_j, \quad \text{and resonance if } \chi_i = \sum_{j=1}^{\ell} s_j \chi_j.$$

We say that a polynomial map $P : \mathcal{E}_x \rightarrow \mathcal{E}_y$ is sub-resonance (resp. resonance) if each component P_i has only terms of sub-resonance (resp. resonance) homogeneous types. We denote by $\mathcal{S}_{x,y}$ (resp. $\mathcal{R}_{x,y}$) the set of all sub-resonance (resp. resonance) polynomials $P : \mathcal{E}_x \rightarrow \mathcal{E}_y$ with $P(0) = 0$ and invertible derivative at 0.

Clearly, for any sub-resonance relation we have $s_j = 0$ for $j < i$ and $\sum s_j \leq \chi_1 / \chi_\ell$. It follows that sub-resonance polynomials have degree at most

$$(3.9) \quad d = d(\chi) = \lfloor \chi_1 / \chi_\ell \rfloor.$$

We will denote $\mathcal{S}_{x,x}$ by \mathcal{S}_x , which is a finite-dimensional Lie group with respect to the composition [GuKt98]. All groups \mathcal{S}_x are isomorphic, moreover, any map $P \in \mathcal{S}_{x,y}$ induces an isomorphism between \mathcal{S}_x and \mathcal{S}_y by conjugation. Any invertible linear map $A : \mathcal{E}_y \rightarrow \mathcal{E}_x$ which respects the splitting induces an isomorphism between the groups \mathcal{S}_x and \mathcal{S}_y . Similar statements hold for the resonance groups $\mathcal{R}_x = \mathcal{R}_{x,x}$.

Note that a linear map is resonance, resp. sub-resonance, if and only if it preserves the splitting (3.4), resp. the associated flag of fast sub-bundles:

$$(3.10) \quad \mathcal{E}_x^1 = \mathcal{V}_x^1 \subset \mathcal{V}_x^2 \subset \cdots \subset \mathcal{V}_x^\ell = \mathcal{E}_x, \quad \text{where } \mathcal{V}_x^i = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^i$$

While the notion of resonance polynomials depends on the splitting, the notion of sub-resonance polynomials depends only on the flag (3.10) and sub-resonance polynomials preserve the flag, see [KS17, Proposition 3.2].

3.4. Narrow spectrum. Now for a given $\chi = (\chi_1, \dots, \chi_\ell)$, where $\chi_1 < \cdots < \chi_\ell < 0$, we will define $\varepsilon_0 = \varepsilon_0(\chi) > 0$ which ensures that the spectrum is sufficiently narrow. Informally, we choose it so that if F has (χ, ε) -spectrum for some $\varepsilon < \varepsilon_0(\chi)$ then all Mather spectrum sub-resonances and resonances of F come from the point spectrum χ and also any non-resonance homogeneous type is contracted by forward or backward iterates of F . This condition is stronger than the narrow band spectrum in [GuKt98].

We define $\tilde{\lambda} < 0$ as the largest value of $-\chi_i + \sum_{j=1}^{\ell} s_j \chi_j$ over all $i \in \{1, \dots, \ell\}$ and non-negative integers s_1, \dots, s_ℓ such that this value is negative, that is, they do not satisfy any sub-resonance relation (3.8):

$$(3.11) \quad \tilde{\lambda} = \max \{ -\chi_i + \sum s_j \chi_j < 0 \} \quad \text{and let } \lambda = \max \{ \tilde{\lambda}, -\chi_1 + (d+1)\chi_\ell \} < 0.$$

The maximum exists since there are at most finitely many values of $-\chi_i + \sum s_j \chi_j$ greater than any given number.

Similarly, we define $\mu < 0$ as the largest value of $\chi_i - \sum_{j=1}^{\ell} s_j \chi_j$ over all $i \in \{1, \dots, \ell\}$ and non-negative integers s_1, \dots, s_{ℓ} such that this value is negative, that is, they satisfy some sub-resonance relation which is not a resonance one (3.8) (we will refer to such homogeneous types as *strict sub-resonance*):

$$(3.12) \quad \mu = \max \left\{ \chi_i - \sum_{j=1}^{\ell} s_j \chi_j < 0 \right\}.$$

The maximum exists since there are at most finitely many sub-resonance relations. Finally, we define

$$(3.13) \quad \varepsilon_0 = \varepsilon_0(\chi) = \min \left\{ -\chi_{\ell}, -\lambda/(d+2), -\mu/(d+1) \right\} > 0.$$

4. STATEMENTS OF RESULTS

First we summarize our basic notations and assumptions.

Assumptions 4.1. *In this section,*

$f : X \rightarrow X$ is a homeomorphism of a compact metric space X ,

\mathcal{E} is a continuous vector bundle over X , equipped with a continuous Riemannian metric,

\mathcal{V} is a neighborhood of zero section in \mathcal{E} and $B_{x,\sigma} \subset \mathcal{V}_x$ for some $\sigma > 0$ and all $x \in X$,

$\mathcal{F} : \mathcal{V} \rightarrow \mathcal{E}$ is a $C^{N,\alpha}$ extension of f (see Def. 3.1) that preserves the zero section,

\mathcal{F} is contracts in the sense $\|\mathcal{F}_x(t)\| \leq \xi \|t\|$ for some $\xi < 1$, and all $x \in X$ and $t \in B_{x,\sigma}$,

$F : \mathcal{E} \rightarrow \mathcal{E}$ is the derivative of \mathcal{F} at the zero section, $F_x = D_0 \mathcal{F}_x : \mathcal{E}_x \rightarrow \mathcal{E}_{f_x}$,

F has (χ, ε) -spectrum (see Def. 3.2) for some $\chi = (\chi_1, \dots, \chi_{\ell})$, with $\chi_1 < \dots < \chi_{\ell} < 0$, and some $\varepsilon < \varepsilon_0 = \varepsilon_0(\chi)$ given by (3.13).

Remark 4.2. *As we outlined in Section 3.2, if F has (χ, ε) -spectrum then there is a continuous Riemannian metric on \mathcal{E} and a continuous orthogonal F -invariant splitting $\mathcal{E} = \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^{\ell}$ such that for each $i = 1, \dots, \ell$ and all $x \in X$ we have*

$$(4.1) \quad e^{\chi_i - \varepsilon} \|t\| \leq \|F_x(t)\| \leq e^{\chi_i + \varepsilon} \|t\| \quad \text{for every } t \in \mathcal{E}_x^i,$$

which also yields (3.6). This is the property that we use in the proof of Theorem 4.3.

We recall that $C^{N,\alpha}$ is the space of C^N functions with N^{th} derivative satisfying α -Hölder condition at 0 (3.1). We will require that the smoothness $N + \alpha$ is higher than the “critical regularity” $\chi_1/\chi_{\ell} \geq 1$. If $N \geq 2$ we allow $\alpha = 0$.

Theorem 4.3 (Normal forms for contracting extensions).

Let \mathcal{F} be an extension of f satisfying Assumptions 4.1. Suppose that $N \in \mathbb{N}$, $0 \leq \alpha \leq 1$,

$$(4.2) \quad \nu = \chi_1 - (N + \alpha)\chi_{\ell} > 0 \quad \text{and} \quad \varepsilon < \nu/(N + \alpha + 1).$$

Then (1) There exists a $C^{N,\alpha}$ coordinate change $\mathcal{H} = \{\mathcal{H}_x\}_{x \in X}$ (see Def. 3.1) with diffeomorphisms $\mathcal{H}_x : B_{x,\sigma} \rightarrow \mathcal{E}_x$ satisfying $\mathcal{H}_x(0) = 0$ and $D_0 \mathcal{H}_x = \text{Id}$ which conjugates \mathcal{F} to a continuous polynomial extension \mathcal{P} of sub-resonance type (see Def. 3.3):

$$(4.3) \quad \mathcal{H}_{f_x} \circ \mathcal{F}_x = \mathcal{P}_x \circ \mathcal{H}_x, \quad \text{where } \mathcal{P}_x \in \mathcal{S}_{x,f_x} \text{ for all } x \in X.$$

(1') There exists a $C^{N,\alpha}$ coordinate change $\mathcal{H}' = \{\mathcal{H}'_x\}_{x \in X}$ with diffeomorphisms $\mathcal{H}'_x : B_{x,\sigma} \rightarrow \mathcal{E}_x$ satisfying $\mathcal{H}'_x(0) = 0$ and $D_0\mathcal{H}'_x = \text{Id}$ which conjugates \mathcal{F} to a continuous polynomial extension \mathcal{P}' of resonance type:

$$(4.4) \quad \mathcal{H}'_{f_x} \circ \mathcal{F}_x = \mathcal{P}'_x \circ \mathcal{H}'_x, \quad \text{where } \mathcal{P}'_x \in \mathcal{R}_{x,f_x} \text{ for all } x \in X.$$

(2) Suppose $\tilde{\mathcal{H}} = \{\tilde{\mathcal{H}}_x\}_{x \in X}$ is another $C^{N,\alpha}$ coordinate change as in (1) conjugating \mathcal{F} to a sub-resonance polynomial extension $\tilde{\mathcal{P}}$. Then there exists a continuous family $\{G_x\}_{x \in X}$ with $G_x \in \mathcal{S}_x$ such that $\mathcal{H}_x = G_x \circ \tilde{\mathcal{H}}_x$. Moreover, if $D_0^{(n)}\tilde{\mathcal{H}}_x = D_0^{(n)}\mathcal{H}_x$ for all $n = 2, \dots, d = \lfloor \chi_1/\chi_\ell \rfloor$, then $\mathcal{H}_x = \tilde{\mathcal{H}}_x$ for all $x \in X$.

(2') Suppose $\tilde{\mathcal{H}}' = \{\tilde{\mathcal{H}}'_x\}_{x \in X}$ is another $C^{N,\alpha}$ coordinate change as in (1') conjugating \mathcal{F} to a resonance polynomial extension $\tilde{\mathcal{P}}'$. Then there exists a continuous family $\{G'_x\}_{x \in X}$ with $G'_x \in \mathcal{R}_x$ such that $\mathcal{H}'_x = G'_x \circ \tilde{\mathcal{H}}'_x$. Moreover, if $D_0^{(n)}\tilde{\mathcal{H}}'_x = D_0^{(n)}\mathcal{H}'_x$ for all $n = 2, \dots, d = \lfloor \chi_1/\chi_\ell \rfloor$, then $\mathcal{H}'_x = \tilde{\mathcal{H}}'_x$ for all $x \in X$.

(3) Let $g : X \rightarrow X$ be a homeomorphism commuting with f and $\mathcal{G} : \mathcal{V} \rightarrow \mathcal{E}$ be a $C^{N',\alpha'}$ extension of g which preserves the zero section and commutes with \mathcal{F} . Suppose that $N' \in \mathbb{N}$ and $0 \leq \alpha' \leq 1$ satisfy $N' \leq N$, $N' + \alpha' \leq N + \alpha$, and

$$(4.5) \quad \nu' = \chi_1 - (N' + \alpha')\chi_\ell > 0 \quad \text{and} \quad \varepsilon < \nu'/(N' + \alpha' + 1).$$

Then the coordinate changes \mathcal{H} and \mathcal{H}' conjugate \mathcal{G} to continuous sub-resonance and resonance polynomial extension respectively, that is

$$(4.6) \quad \mathcal{H}_{g_x} \circ \mathcal{G}_x \circ \mathcal{H}_x^{-1} \in \mathcal{S}_{x,f_x} \quad \text{and} \quad \mathcal{H}'_{g_x} \circ \mathcal{G}_x \circ (\mathcal{H}'_x)^{-1} \in \mathcal{R}_{x,f_x} \quad \text{for all } x \in X.$$

In particular, \mathcal{G} is a $C^{N,\alpha}$ extension

Corollary 4.4. Suppose that \mathcal{F} in the theorem is a C^∞ extension. Then the coordinate changes \mathcal{H} in part (1) and \mathcal{H}' in part (1') are also C^∞ .

Remark 4.5 (Global version). Suppose that $\mathcal{F} : \mathcal{E} \rightarrow \mathcal{E}$ is a globally defined extension which satisfies the assumptions of Theorem 4.3 and either contracts fibers or, more generally, satisfies the property that for any compact set $K \subset \mathcal{V}$ and any neighborhood U of the zero section we have $\mathcal{F}^n(K) \subset U$ for all sufficiently large n . Then the coordinate changes \mathcal{H} and \mathcal{H}' can be uniquely extended "by invariance" $\mathcal{H}_x = (P_x^n)^{-1} \circ \mathcal{H}_{f^n(x)} \circ \mathcal{F}_x^n$ to the family of global $C^{N,\alpha}$ coordinate changes with diffeomorphisms $\mathcal{H}_x, \mathcal{H}'_x : \mathcal{E}_x \rightarrow \mathcal{E}_x$ satisfying (4.3) and (4.4) respectively. Moreover, if the extension \mathcal{G} as in (3) is also globally defined, then it satisfies (4.6) globally.

4.1. Normal forms for contracting foliations. Now we apply the results above in the context of diffeomorphisms with invariant contracting foliation.

Let f be a C^r diffeomorphism of a compact manifold X . We will consider $r > 1$, and for $r \notin \mathbb{N}$ we will understand C^r in the usual sense that the derivative of order $N = \lfloor r \rfloor$ is Hölder with exponent $\alpha = r - \lfloor r \rfloor$. We will consider an f -invariant continuous foliation \mathcal{W} of X with *uniformly C^r leaves*, by which we mean that for some $R > 0$ the balls $B^{\mathcal{W}}(x, R)$ of radius R in the intrinsic Riemannian metric of the leaf can be given

by C^r embeddings which depend continuously on x in C^N topology and, if $r \notin \mathbb{N}$, have α -Hölder derivative of order N with uniformly bounded Hölder constant. Similarly, for such a foliation we will say that a function g is *uniformly C^r along \mathcal{W}* if its restrictions to $B^{\mathcal{W}}(x, R)$ depend continuously on x in C^N topology and have α -Hölder derivative of order N with uniformly bounded Hölder constant. We also allow $r = \infty$, in which case uniformly C^∞ means uniformly C^N for each N .

Theorem 4.6 (Normal forms for contracting foliations). *Let f be a C^r , $r \in (1, \infty]$, diffeomorphism of a smooth compact manifold X , and let \mathcal{W} be an f -invariant topological foliation of X with uniformly C^r leaves. Suppose that the linear extension $F = Df|_{T\mathcal{W}}$ has (χ, ε) -spectrum (or alternatively F satisfies the condition in Remark 4.2) for some $\chi = (\chi_1, \dots, \chi_\ell)$, where $\chi_1 < \dots < \chi_\ell < 0$, and some $\varepsilon < \varepsilon_0 = \varepsilon_0(\chi)$ given by (3.13). Suppose also that $r > \chi_1/\chi_\ell$ and*

$$(4.7) \quad \varepsilon < \nu/(r+1), \quad \text{where} \quad \nu = \chi_1 - r\chi_\ell > 0.$$

Then (1) There exists a family $\{\mathcal{H}_x\}_{x \in X}$ of C^r diffeomorphisms $\mathcal{H}_x : \mathcal{W}_x \rightarrow T_x\mathcal{W}$ satisfying $\mathcal{H}_x(0) = 0$ and $D_0\mathcal{H}_x = Id$ such that for each $x \in X$,

$$\mathcal{P}_x = \mathcal{H}_{f(x)} \circ f \circ \mathcal{H}_x^{-1} : T_x\mathcal{W} \rightarrow T_{f(x)}\mathcal{W} \text{ is in } \mathcal{S}_{x,fx}.$$

The maps $\mathcal{H}_x|_{B^{\mathcal{W}}(x,R)}$ depend continuously on $x \in X$ in C^N topology with $N = \lfloor r \rfloor$ and, if $\alpha = r - N > 0$, they have α -Hölder derivative of order N with uniformly bounded Hölder constant.

(1') There exists a family $\{\mathcal{H}'_x\}_{x \in X}$ of diffeomorphisms as in (1) so that for all $x \in X$,

$$\mathcal{P}_x = \mathcal{H}'_{f(x)} \circ f \circ (\mathcal{H}'_x)^{-1} : T_x\mathcal{W} \rightarrow T_{f(x)}\mathcal{W} \text{ is in } \mathcal{R}_{x,fx}.$$

(2) Suppose $\tilde{\mathcal{H}} = \{\tilde{\mathcal{H}}_x\}_{x \in X}$ is another family of diffeomorphisms as in (1) conjugating $f|_{\mathcal{W}}$ to sub-resonance polynomials $\tilde{\mathcal{P}}_x \in \mathcal{S}_{x,fx}$. Then there exists a continuous family $\{G_x\}_{x \in X}$ with $G_x \in \mathcal{S}_x$ such that $\mathcal{H}_x = G_x \circ \tilde{\mathcal{H}}_x$. Moreover, if $D_0^{(n)}\tilde{\mathcal{H}}_x = D_0^{(n)}\mathcal{H}_x$ for all $n = 2, \dots, d = \lfloor \chi_1/\chi_\ell \rfloor$, then $\mathcal{H}_x = \tilde{\mathcal{H}}_x$ for all $x \in X$.

(2') Suppose $\tilde{\mathcal{H}}' = \{\tilde{\mathcal{H}}'_x\}_{x \in X}$ is another family of diffeomorphisms as in (1') conjugating $f|_{\mathcal{W}}$ to resonance polynomials $\tilde{\mathcal{P}}_x \in \mathcal{R}_{x,fx}$. Then there exists a continuous family $\{G'_x\}_{x \in X}$ with $G'_x \in \mathcal{R}_x$ such that $\mathcal{H}'_x = G'_x \circ \tilde{\mathcal{H}}'_x$. Moreover, if $D_0^{(n)}\tilde{\mathcal{H}}'_x = D_0^{(n)}\mathcal{H}'_x$ for all $n = 2, \dots, d = \lfloor \chi_1/\chi_\ell \rfloor$, then $\mathcal{H}'_x = \tilde{\mathcal{H}}'_x$ for all $x \in X$.

(3) Let g be a homeomorphism of X which commutes with f , preserves \mathcal{W} , and is uniformly $C^{r'}$ along the leaves of \mathcal{W} . Suppose that $1 < r' \leq r$ satisfies

$$(4.8) \quad \nu' = \chi_1 - r'\chi_\ell > 0 \quad \text{and} \quad \varepsilon < \nu'/(r'+1).$$

Then $Q_x = \mathcal{H}_{g(x)} \circ g \circ \mathcal{H}_x^{-1} \in \mathcal{S}_{x,gx}$ and $Q'_x = \mathcal{H}'_{g(x)} \circ g \circ (\mathcal{H}'_x)^{-1} \in \mathcal{R}_{x,gx}$ for all $x \in X$. In particular, \mathcal{G}_x is uniformly C^r along the leaves of \mathcal{W} .

(4) For any $x \in X$ and $y \in \mathcal{W}_x$, the maps $\mathcal{H}_y \circ \mathcal{H}_x^{-1}$ and $\mathcal{H}'_y \circ (\mathcal{H}'_x)^{-1}$ from $T_x\mathcal{W}$ to $T_y\mathcal{W}$ are compositions of a sub-resonance polynomial in $\mathcal{S}_{x,y}$ with a translation.

(5) The family $\{\mathcal{H}_x\}_{x \in X}$ as in (1) can be chosen so that \mathcal{H}_x which depends $C^{\lfloor r \rfloor}$ on x along the leaves of \mathcal{W} .

We note that there is no analog of (5) for the resonance case. Also, the transition maps $\mathcal{H}'_y \circ (\mathcal{H}'_x)^{-1}$ in (4) are only sub-resonance, so this is just a particular case of the result for $\mathcal{H}_y \circ (\mathcal{H}_x)^{-1}$.

Another way to interpret (4) is to view \mathcal{H}_x as a coordinate chart on \mathcal{W}_x , identifying it with $T_x\mathcal{W}$, and in particular identifying $T_y\mathcal{W}$ with $T_{\mathcal{H}_x(y)}(T_x\mathcal{W})$ by $D_y\mathcal{H}_x$. In this coordinate chart, (4) yields that all transition maps $\mathcal{H}_y \circ \mathcal{H}_x^{-1}$ for $y \in \mathcal{W}_x$ are in the group $\bar{\mathcal{S}}_x$ generated by \mathcal{S}_x and the translations of $T_x\mathcal{W}$. Clearly, this is a finite dimensional Lie group which acts transitively on $T_x\mathcal{W}$.

Remark 4.7. The diffeomorphism \mathcal{H}_x maps the sub-foliations of \mathcal{W}_x by fast leaves tangent to the fast flag (3.10) to the linear sub-foliations of $T_x\mathcal{W}$ by subspaces parallel to the flag, and the compositions $\mathcal{H}_y \circ \mathcal{H}_x^{-1}$ map flag to flag, see [KS16, Section 3.2]. It follows that these fast sub-foliations of \mathcal{W}_x are as smooth inside \mathcal{W}_x as \mathcal{H}_x . While smoothness of fast sub-foliations is a well-known phenomenon, normal form results give an alternative proof.

Corollary 4.8. Under the assumptions of the Theorem 4.6, if $\ell = 1$, i.e. $\chi_1 = \chi_\ell = \chi$, then $\mathcal{P}_x = Df|_{T_x\mathcal{W}}$ in (1) and $Q_x = Dg|_{T_x\mathcal{W}}$ in (3) are linear normal forms, the family $\{\mathcal{H}_x\}_{x \in X}$ as in (1) is unique, the maps $\mathcal{H}_y \circ \mathcal{H}_x^{-1} : \mathcal{E}_x \rightarrow \mathcal{E}_y$ are affine for all $x \in X$ and $y \in \mathcal{W}_x$, and \mathcal{H}_y depends $C^{\lfloor r \rfloor}$ -smoothly on y along the leaves of \mathcal{W} .

In this case we can take $\varepsilon < -\alpha\chi/(2+\alpha)$, where $\alpha = \min\{1, r-1\}$. This means that $Df|_{T\mathcal{W}}$ is a contraction whose characteristic set is contained in an open interval with ratio of endpoints at most $1 + \alpha$. This is the ‘‘constant rate’’ version of $1/2$ pinching (2.1) (cf. Remark 2.2).

5. PROOF OF THEOREM 4.3 AND COROLLARY 4.4

We begin with the proof of part (1). First we construct Taylor polynomials at zero of degree N for \mathcal{H}_x and the corresponding terms of sub-resonance polynomials \mathcal{P}_x . Since (4.2) implies $N \geq d = \lfloor \chi_1/\chi_\ell \rfloor$, this will fully define \mathcal{P}_x , whose degree is at most d .

5.1. Construction of \mathcal{P} and of the Taylor polynomial for \mathcal{H} . For each $x \in X$ and map $\mathcal{F}_x : \mathcal{E}_x \rightarrow \mathcal{E}_{f_x}$ we consider its Taylor polynomial of degree N at $t = 0$:

$$(5.1) \quad \mathcal{F}_x(t) \sim \sum_{n=1}^N F_x^{(n)}(t).$$

As a function of t , $F_x^{(n)}(t) : \mathcal{E}_x \rightarrow \mathcal{E}_{f_x}$ is a homogeneous polynomial map of degree n . We will view the family $F^{(n)} = \{F_x^{(n)}\}_{x \in X}$ as a section of the corresponding bundle of homogeneous polynomials. We will use similar notations for the Taylor polynomials

at $t = 0$ of the desired coordinate change $\mathcal{H}_x(t)$ and the corresponding sub-resonance polynomial extension $\mathcal{P}_x(t)$:

$$\mathcal{H}_x(t) \sim \sum_{n=1}^N H_x^{(n)}(t) \quad \text{and} \quad \mathcal{P}_x(t) = \sum_{n=1}^d P_x^{(n)}(t).$$

We will inductively construct the Taylor terms $H^{(n)} = \{H_x^{(n)}\}_{x \in X}$ and $P^{(n)} = \{P_x^{(n)}\}_{x \in X}$ as continuous sections of the corresponding bundle of homogeneous polynomials.

For the first derivative we define

$$H_x^{(1)} = \text{Id} : \mathcal{E}_x \rightarrow \mathcal{E}_x \quad \text{and} \quad P_x^{(1)} = F_x \quad \text{for all } x \in X.$$

Now we assume that the terms of order less than n are constructed. Using these linear terms in the conjugacy equation $\mathcal{H}_{f_x} \circ \mathcal{F}_x = \mathcal{P}_x \circ \mathcal{H}_x$ we write

$$\left(\text{Id} + \sum_{i=2}^N H_{f_x}^{(i)} \right) \circ \left(F_x + \sum_{i=2}^N F_x^{(i)} \right) \sim \left(F_x + \sum_{i=2}^d P_x^{(i)} \right) \circ \left(\text{Id} + \sum_{i=2}^N H_x^{(i)} \right).$$

and considering the terms of degree n , $2 \leq n \leq N$, we obtain

$$F_x^{(n)} + H_{f_x}^{(n)} \circ F_x + \sum H_{f_x}^{(i)} \circ F_x^{(j)} = F_x \circ H_x^{(n)} + P_x^{(n)} + \sum P_x^{(j)} \circ H_x^{(i)},$$

where the summations are over all i and j such that $ij = n$ and $1 < i, j < n$. We rewrite the equation as

$$(5.2) \quad F_x^{-1} \circ P_x^{(n)} = -H_x^{(n)} + F_x^{-1} \circ H_{f_x}^{(n)} \circ F_x + Q_x,$$

where

$$(5.3) \quad Q_x = F_x^{-1} \left(F_x^{(n)} + \sum_{ij=n, 1 < i, j < n} H_{f_x}^{(i)} \circ F_x^{(j)} - P_x^{(j)} \circ H_x^{(i)} \right).$$

We note that Q_x is composed only of terms $H^{(i)}$ and $P^{(i)}$ with $1 < i < n$, which are already constructed, and terms $F^{(i)}$ with $1 < i \leq n$, which are given. Thus by the inductive assumption Q_x is defined for all $x \in X$ and is continuous in x .

Let $\mathcal{Q}_x^{(n)}$ be the space of all homogeneous polynomial maps on \mathcal{E}_x of degree n , and let $\mathcal{S}_x^{(n)}$ and $\mathcal{N}_x^{(n)}$ be the subspaces of sub-resonance and non sub-resonance polynomials respectively. We seek $H_x^{(n)}$ so that the right side of (5.2) is in $\mathcal{S}_x^{(n)}$, and hence so is $P_x^{(n)}$ when defined by this equation.

Projecting (5.2) to the factor bundle $\mathcal{Q}^{(n)}/\mathcal{S}^{(n)}$, our goal is to solve the equation

$$(5.4) \quad 0 = -\bar{H}_x^{(n)} + F_x^{-1} \circ \bar{H}_{f_x}^{(n)} \circ F_x + \bar{Q}_x,$$

where $\bar{H}^{(n)}$ and \bar{Q} are the projections of $H^{(n)}$ and Q respectively.

We consider the bundle automorphism $\Phi : \mathcal{Q}^{(n)} \rightarrow \mathcal{Q}^{(n)}$ covering $f^{-1} : X \rightarrow X$ given by the maps $\Phi_x : \mathcal{Q}_{f_x}^{(n)} \rightarrow \mathcal{Q}_x^{(n)}$

$$(5.5) \quad \Phi_x(R) = F_x^{-1} \circ R \circ F_x.$$

Since F preserves the splitting $\mathcal{E} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^\ell$, it follows from the definition that the sub-bundles $\mathcal{S}^{(n)}$ and $\mathcal{N}^{(n)}$ are $\bar{\Phi}$ -invariant. We denote by $\tilde{\Phi}$ the induced automorphism of $\mathcal{Q}^{(n)}/\mathcal{S}^{(n)}$ and conclude that (5.4) is equivalent to

$$(5.6) \quad \bar{H}_x^{(n)} = \tilde{\Phi}_x(\bar{H}_{f_x}^{(n)}), \quad \text{where } \tilde{\Phi}_x(R) = \bar{\Phi}_x(R) + \bar{Q}_x.$$

Thus a solution of (5.4) is a $\tilde{\Phi}$ -invariant section of $\mathcal{Q}^{(n)}/\mathcal{S}^{(n)}$. In Lemma 5.2 below we will show that $\tilde{\Phi}$ is a contraction and hence has a unique continuous invariant section, which can be explicitly written as

$$(5.7) \quad \bar{H}_x = \sum_{k=0}^{\infty} (F_x^k)^{-1} \circ \bar{Q}_{f^k x} \circ F_x^k, \quad \text{where } F_x^k = F_{f^{k-1}x} \circ \cdots \circ F_{f_x} \circ F_x.$$

We equipped \mathcal{E} with a continuous Riemannian metric as in Remark 4.2 for which the splitting $\mathcal{E} = \mathcal{E}^1 \oplus \cdots \oplus \mathcal{E}^\ell$ is orthogonal and so that (4.1) and hence (3.6) hold.

The norm of a homogeneous polynomial map $R : \mathcal{E}_x \rightarrow \mathcal{E}_y$ of degree n is defined as

$$(5.8) \quad \|R\| = \sup\{\|R(v)\| : v \in E, \|v\| = 1\}.$$

It follows that for any other homogeneous polynomial map $P : \mathcal{E}_z \rightarrow \mathcal{E}_x$ we have

$$(5.9) \quad \|R \circ P\| \leq \|R\| \cdot \|P\|^n.$$

First, we look at the action of Φ on polynomials of specific homogeneous type.

Lemma 5.1. *Let $Q \in \mathcal{Q}_x^{(n)}$ and $R \in \mathcal{Q}_{f_x}^{(n)}$ be polynomials of homogeneous type $s = (s_1, \dots, s_\ell)$ with $s_1 + \cdots + s_\ell = n$. Then*

$$(5.10) \quad \|\Phi_x(R)\| \leq e^{-\chi_i + \sum s_j \chi_j + (n+1)\varepsilon} \|R\| \quad \text{and} \quad \|\Phi_x^{-1}(Q)\| \leq e^{\chi_i - \sum s_j \chi_j + (n+1)\varepsilon} \|R\|.$$

Proof. We will prove the first inequality, the second one is obtained similarly. Suppose that $v = v_1 + \cdots + v_\ell$, where $v_j \in \mathcal{E}_x^j$, and $\|v\|_x = 1$. We denote $a_j = \|F|_{\mathcal{E}_x^j}\|$ and observe that $F_x(v_j) = a_j v'_j \in \mathcal{E}_{f_x}^j$ with $\|v'_j\| \leq \|v_j\|$. Since R has homogeneous type $s = (s_1, \dots, s_\ell)$ we obtain by (3.7) that

$$(5.11) \quad (R \circ F_x)(v) = R(a_1 v'_1 + \cdots + a_\ell v'_\ell) = a_1^{s_1} \cdots a_\ell^{s_\ell} \cdot R(v'_1 + \cdots + v'_\ell).$$

where $v' = v'_1 + \cdots + v'_\ell$ has $\|v'\|_{f_x} \leq \|v\|_x = 1$ by orthogonality of the splitting. Thus

$$\|(R \circ F_x)(v)\| = a_1^{s_1} \cdots a_\ell^{s_\ell} \cdot \|R(v')\| \leq a_1^{s_1} \cdots a_\ell^{s_\ell} \cdot \|R\|$$

for any $v \in \mathcal{E}_x$ with $\|v\|_x = 1$, so by definition (5.8) we obtain $\|R \circ F_x\| \leq a_1^{s_1} \cdots a_\ell^{s_\ell} \cdot \|R\|$. Now (5.9) yields

$$\begin{aligned} \|\Phi_x(R)\| &= \|F|_{\mathcal{E}_x^i}^{-1} \circ R \circ F_x\| \leq \|F|_{\mathcal{E}_x^i}^{-1}\| \cdot \|R \circ F_x\| \leq \\ &\leq \|F|_{\mathcal{E}_x^i}^{-1}\| \cdot a_1^{s_1} \cdots a_\ell^{s_\ell} \cdot \|R\| \leq e^{-\chi_i + \varepsilon} \cdot \prod_j (e^{\chi_j + \varepsilon})^{s_j} \cdot \|R\|. \end{aligned}$$

Since $a_j = \|F|_{\mathcal{E}_x^j}\| \leq e^{\chi_j + \varepsilon}$ and $\|F|_{\mathcal{E}_x^i}^{-1}\| \leq e^{-\chi_i + \varepsilon}$ by (3.6). \square

Lemma 5.2. *The map $\Phi : \mathcal{N}^{(n)} \rightarrow \mathcal{N}^{(n)}$ given by (5.5) is a contraction over f^{-1} , and hence so is $\tilde{\Phi} : \mathcal{Q}^{(n)}/\mathcal{S}^{(n)} \rightarrow \mathcal{Q}^{(n)}/\mathcal{S}^{(n)}$ given by (5.6). More precisely, $\|\Phi_x(R)\| \leq e^{\lambda+(d+2)\varepsilon} \cdot \|R\|$.*

Proof. The statement about $\tilde{\Phi}$ follows since the linear part $\bar{\Phi}$ of $\tilde{\Phi}$ is given by Φ when $\mathcal{Q}^{(n)}/\mathcal{S}^{(n)}$ is naturally identified with $\mathcal{N}^{(n)}$.

For all non sub-resonance homogeneous types we have $-\chi_i + \sum s_j \chi_j \leq \tilde{\lambda}$ by the definition of $\tilde{\lambda}$ (3.11) and hence for any $R \in \mathcal{N}_{fx}^{(n)}$ Lemma 5.1 yields the estimate $\|\Phi_x(R)\| \leq e^{\tilde{\lambda}+(n+1)\varepsilon} \cdot \|R\|$. For all $n \leq d$ the exponent satisfies $\tilde{\lambda} + (n+1)\varepsilon \leq \lambda + (d+1)\varepsilon < 0$ since $\varepsilon < \varepsilon_0$ given by (3.13).

If $d+1 \leq n \leq N$, then $\mathcal{S}^{(n)} = 0$ and $\mathcal{Q}^{(n)} = \mathcal{N}^{(n)}$. In this case for any $R \in \mathcal{Q}_{fx}^{(n)}$ we can estimate $\|\Phi_x(R)\| \leq e^{-\chi_1+n\chi_\ell+(n+1)\varepsilon} \cdot \|R\|$ and by the definition of λ (3.11) we have $-\chi_1 + (d+1)\chi_\ell \leq \lambda < 0$ and hence the exponent satisfies

$$-\chi_1 + n\chi_\ell + (n+1)\varepsilon \leq \lambda + (d+2)\varepsilon + (n - (d+1))(\varepsilon + \chi_\ell) < 0$$

since $\varepsilon < \varepsilon_0$ given by (3.13). \square

We conclude that the unique continuous invariant section (5.7) for $\tilde{\Phi}$ is the unique continuous solution $\bar{H}^{(n)}$ of (5.4). Now can we choose a continuous section $H^{(n)}$ of $\mathcal{Q}^{(n)}$ which projects to $\bar{H}^{(n)}$. Such $H^{(n)}$ is defined uniquely up to a continuous section $\mathcal{S}^{(n)}$.

Remark 5.3. *For example one can take $H^{(n)}$ in $\mathcal{N}^{(n)}$. However, in the foliation setting the bundles $\mathcal{Q}^{(n)}$, $\mathcal{S}^{(n)}$, their quotient, and $\bar{H}^{(n)}$ are often more regular along the leaves than $\mathcal{N}^{(n)}$. In this case one can make a more regular choice for $H^{(n)}$ leading to better dependence of \mathcal{H}_x on x along the leaves. See [KS16] for more details of this argument.*

Once a specific lift $H^{(n)}$ is chosen, $P_x^{(n)}$ is uniquely define by equation (5.2) and is a continuous section $P^{(n)}$. This completes the inductive step and the construction of $H^{(n)}$ and $P^{(n)}$, $n = 1, \dots, N$.

Thus we have constructed the N -th Taylor polynomials for the coordinate changes

$$(5.12) \quad \mathcal{H}_x^N(t) = \sum_{n=1}^N H_x^{(n)}(t) \quad \text{of degree } N \geq d = \lfloor \chi_1/\chi_\ell \rfloor$$

and the polynomial maps $\mathcal{P}_x(t) = \sum_{n=1}^d P_x^{(n)}(t)$.

5.2. Construction of the coordinate change \mathcal{H} . In this section we complete the proof of part (1) by constructing the actual coordinate changes \mathcal{H}_x with the Taylor polynomial \mathcal{H}_x^N given by (5.12). To simplify the calculations we note that $\mathcal{H}_x^N(t)$ is a diffeomorphism on some neighborhood $\tilde{\mathcal{V}}_x$ of $0 \in \mathcal{E}_x$ since its differential at 0 is Id, moreover the size of $\tilde{\mathcal{V}}_x$ can be bounded away from 0 by compactness of X . Thus we can consider extension $\tilde{\mathcal{F}}_x(t) = \mathcal{H}_{fx}^N \circ \mathcal{F}_x \circ (\mathcal{H}_x^N)^{-1}$. By the construction of \mathcal{H}^N , the maps $\tilde{\mathcal{F}}_x$ and \mathcal{P}_x have the same derivatives at $t = 0$ up to order N for each $x \in X$.

Since the construction is done in a sufficiently small neighborhood of zero section, we can replace \mathcal{F} by $\tilde{\mathcal{F}}_x$, and so henceforth we assume that \mathcal{F} itself has this property.

We rewrite the conjugacy equation $\mathcal{H}_{f_x} \circ \mathcal{F}_x = \mathcal{P}_x \circ \mathcal{H}_x$ in the form

$$(5.13) \quad \mathcal{H}_x = \tilde{T}(\mathcal{H})_x = \mathcal{P}_x^{-1} \circ \mathcal{H}_{f_x} \circ \mathcal{F}_x.$$

so that solution $\mathcal{H} = \{\mathcal{H}_x\}$ is a fixed point of the operator \tilde{T} . Since \mathcal{P}_x is a sub-resonance polynomial with invertible linear part $\mathcal{P}_x^{(1)} = F_x$, by the group property the inverse \mathcal{P}_x^{-1} is also a sub-resonance polynomial and thus has degree at most $d \leq N$.

Denoting $\bar{H}_x = \mathcal{H}_x - \text{Id}$ we rewrite (5.13) as

$$(5.14) \quad \bar{H}_x = T(\bar{H})_x = \mathcal{P}_x^{-1} \circ (\text{Id} + \bar{H}_{f_x}) \circ \mathcal{F}_x - \text{Id}.$$

Thus the coordinate change \mathcal{H} corresponds to the fixed point $\bar{H} = T(\bar{H})$ of the operator T on the space of continuous families of smooth functions. We will show that T is a contraction on an appropriate space and thus has a unique fixed point.

For any $x \in X$ we consider the ball $B_{x,r}$ in \mathcal{E}_x centered at 0 of radius r and denote

$$\mathcal{C}_x = \mathcal{C}_{x,r} = \{R \in C^{N,\alpha}(B_{x,r}, \mathcal{E}_x) : D_0^{(k)}R = 0, k = 0, \dots, N\},$$

where $C^{N,\alpha}(B_{x,r}, \mathcal{E}_x)$ and its norm are defined as in (3.2). We note that for any $R \in \mathcal{C}_x$ the α -Hölder constant (3.1) of $D^{(N)}R$ at 0 is

$$(5.15) \quad \|D^{(N)}R\|_\alpha = \sup \{\|D_t^{(N)}R\| \cdot \|t\|^{-\alpha} : 0 \neq t \in B_{x,r}\}.$$

For any $R \in \mathcal{C}_x$ lower derivatives can be estimated by the mean value theorem as

$$(5.16) \quad \|D_t^{(n)}R\| \leq \|t\|^{N-n} \cdot \sup \{\|D_s^{(N)}R\| : \|s\| \leq \|t\|\},$$

so using the above Hölder constant we obtain that for any $0 \leq n < N$ and $t \in B_{x,r}$,

$$(5.17) \quad \|D_t^{(n)}R\| \leq \|t\|^{1+\alpha} \cdot \|D^{(N)}R\|_\alpha.$$

Thus for $r < 1$ the norms of all derivatives are dominated by the Hölder constant and hence

$$(5.18) \quad \|R\|_{C^{N,\alpha}(B_{x,r}, \mathcal{E}_x)} = \|D^{(N)}R\|_\alpha.$$

It follows that \mathcal{C}_x equipped with the norm $\|D^{(N)}R\|_\alpha$ is a Banach space. We denote by \mathcal{C} the bundle over X with fibers \mathcal{C}_x and by \mathcal{B} the space of sections $\bar{R} = \{R_x\}_{x \in X}$ of \mathcal{C} which are bounded in $C^{N,\alpha}$ norm and continuous in C^N norm. Then \mathcal{B} is a Banach space with the norm $\|\bar{R}\|_{\mathcal{C}} = \sup_x \|R_x\|_\alpha$.

We consider T as an operator on \mathcal{B} . It follows from the definition of \mathcal{C}_x and the coincidence of the derivatives of \mathcal{P}_x and \mathcal{F}_x at 0 that $T(\bar{R})$ is in \mathcal{B} . We will show that that, for a sufficiently small r , T is a contraction on some ball B_γ in \mathcal{B} .

We will now define the parameters for this argument. First we note that $0 < \varepsilon < -\chi_\ell$ since $\varepsilon < \varepsilon_0$ given by (3.13), and that we have $\nu - (N + 1 + \alpha)\varepsilon > 0$ by assumption (4.2) so we can take $\varepsilon' > 0$ satisfying

$$(5.19) \quad \chi_\ell + \varepsilon + \varepsilon' < 0 \quad \text{and} \quad \delta = \nu - (N + 1 + \alpha)(\varepsilon + \varepsilon') > 0.$$

We also recall that by (3.6) we have

$$(5.20) \quad D_0 \mathcal{P}_x = D_0 \mathcal{F}_x = F_x, \quad \|F_x\| \leq e^{\chi_\ell + \varepsilon}, \quad \text{and} \quad \|F_x^{-1}\| \leq e^{-\chi_1 + \varepsilon}$$

Now we can choose $\rho < \min\{1, \sigma\}$ sufficiently small so that for all $x \in X$ we have

$$(5.21) \quad \|D_t \mathcal{F}_x\| \leq e^{\chi_\ell + \varepsilon + \varepsilon'} \quad \text{and} \quad \|D_t (\mathcal{P}_x)^{-1}\| \leq e^{-\chi_1 + \varepsilon + \varepsilon'} \quad \text{for all } t \in B_{x, \rho},$$

so in particular $\mathcal{F}_x : B_{x, \rho} \rightarrow B_{f_x, \rho}$ is a contraction. We choose K so that

$$(5.22) \quad \|\mathcal{F}\|_{C^N(B_{x, \rho})} \leq K \quad \text{and} \quad \|(\mathcal{P}_x)^{-1}\|_{C^N(B_{x, \rho})} \leq K \quad \text{for all } x \in X.$$

Since δ given by (5.19) is positive, we can define $\theta > 0$ by

$$(5.23) \quad 1 - 2\theta = e^{-\delta} < 1 \quad \text{and let } \gamma = \max\{1, \|T(\bar{0})\|_{\mathcal{B}}/\theta\},$$

here, as r is not yet defined, we take $r = \rho$ in the definition of the norm $\|T(\bar{0})\|_{\mathcal{B}}$ (this does not create problems since the norm decreases with r). To show that T is a contraction on the ball B_γ in \mathcal{B} centered at $\bar{0}$ of radius γ we will estimate the norm of its differential by $1 - \theta$. For this we choose $r > 0$ satisfying

$$(5.24) \quad r < \rho < 1, \quad r < \rho/(1 + \gamma), \quad r \leq \theta/(c_3(K, N) \gamma^N)$$

where constant $c_3(K, N)$ from (5.31) depends only on N and K .

Now we will calculate the differential of T on B_γ and estimate its norm. For any $\bar{R}, \bar{S} \in B_\gamma$ we can write

$$(T(\bar{R} + \bar{S}) - T(\bar{R}))_x = (\mathcal{P}_x)^{-1} \circ (\text{Id} + R_{f_x} + S_{f_x}) \circ \mathcal{F}_x - (\mathcal{P}_x)^{-1} \circ (\text{Id} + R_{f_x}) \circ \mathcal{F}_x.$$

Differentiating $(\mathcal{P}_x)^{-1}$ and denoting

$$y(t) = (\text{Id} + R_{f_x})(\mathcal{F}_x(t)) = \mathcal{F}_x(t) + R_{f_x}(\mathcal{F}_x(t)) \quad \text{and} \quad z(t) = S_{f_x}(\mathcal{F}_x(t))$$

we obtain

$$(T(\bar{R} + \bar{S}) - T(\bar{R}))_x(t) = D_{y(t)}(\mathcal{P}_x)^{-1} z(t) + E(z(t)),$$

where E is a polynomial with terms of degree at least two. It follows that $\|E(z(t))\|_{\mathcal{C}} = O(\|\bar{S}\|_{\mathcal{C}}^2)$ and so the differential of T is given by

$$([D_{\bar{R}} T] \bar{S})_x(t) = D_{y(t)}(\mathcal{P}_x)^{-1} S_{f_x}(\mathcal{F}_x(t)) = A_x(y(t))z(t),$$

where $A_x(s) = D_s(\mathcal{P}_x)^{-1}$. To estimate the norm we consider the derivative of order N .

Since $A_x(y(t))$ is a linear operator on z , the product rule yields

$$(5.25) \quad D^{(N)}[A_x(y(t))z(t)] = A_x(y(t))D^{(N)}z(t) + \sum c_{m,l} D^{(m)}A_x(y(t))D^{(l)}z(t),$$

where $m + l = N$ and $l < N$ for all terms in the sum. Differentiating $z(t)$ we get

$$D^{(l)}z(t) = D^{(l)}S_{f_x}(\mathcal{F}_x(t)) = \sum D_{t'}^{(i)} S_{f_x} \circ D_t^{(j)} \mathcal{F}_{x_k},$$

where $ij = l$ and $t' = \mathcal{F}_x(t)$. Only the first term in (5.25) contains $D^{(N)}S_{f_x}$ so

$$(5.26) \quad D_t^{(N)}([D_{\bar{R}} T] \bar{S})_x = D_{y(t)}^{(1)} (\mathcal{P}_{x_k})^{-1} \circ D_{t'}^{(N)} S_{f_x} \circ D_t^{(1)} \mathcal{F}_x + J_x,$$

where J_x consists of a fixed number $c(N)$ of terms of the type

$$D_t^{(m)} A_x(y(t)) \left(D_{t'}^{(i)} S_{f_x} \circ D_t^{(j)} \mathcal{F}_x \right), \quad i < N, \quad m + ij = N,$$

whose norms can be estimated by

$$(5.27) \quad \|A_x(y(t))\|_{C^N} \cdot \|D_{t'}^{(i)} S_{f_x}\| \cdot \|\mathcal{F}_x\|_{C^N}^{N-1}.$$

We start with the middle term and observe that

$$(5.28) \quad \|t'\| = \|\mathcal{F}_x(t)\| \leq e^{\chi_\ell + \varepsilon + \varepsilon'} \|t\| < \|t\| \leq r$$

by (5.21). Since $i < N$, we can estimate the middle term using (5.17)

$$(5.29) \quad \|D_{t'}^{(i)} S_{f_x}\| \leq \|t'\|^{1+\alpha} \cdot \|D^{(N)} S_{f_x}\|_\alpha < \|t\|^{1+\alpha} \cdot \|\bar{S}\|_c \leq r \|t\|^\alpha \cdot \|\bar{S}\|_c.$$

For the first term we note that $t'' = y(t) \in B_{x,\rho}$, indeed since $\|\bar{R}\|_c \leq \gamma$ we get

$$(5.30) \quad \|t''\| = \|y(t)\| = \|(\text{Id} + R_{f_x})(\mathcal{F}_x(t))\| = \|t' + R_{f_x}(t')\| < (r + \gamma r) < \rho$$

by (5.28) and the choice of r (5.24). We also use an estimate for the norm of composition of smooth maps, see e.g [dlLO98]:

Lemma 5.4. [dlLO98, Theorem 4.3(ii.3)] *For any $N \geq 1$ there exist a constant M_N such that $\|h \circ g\|_{C^N} \leq M_N \|h\|_{C^N} (1 + \|g\|_{C^N})^N$ for any $h, g \in C^N(B_{x,\rho})$.*

Using this together with (5.22), $\|\bar{R}\|_c \leq \gamma$, and $\gamma \geq 1$ and we can write

$$\|y(t)\|_{C^N} = \|(\text{Id} + R_{f_x}) \circ \mathcal{F}_x\|_{C^N} \leq M_N (1 + \gamma) (1 + K)^N \leq c_1(K, N) \gamma,$$

where $c_1(K, N) = 2M_N(1 + K)^N$. Now we can estimate the first term in (5.27)

$$\|A_x(y(t))\|_{C^N} \leq M_N \|(\mathcal{P}_x)^{-1}\|_{C^N} \|y(t)\|_{C^N}^N \leq M_N K [1 + c_1(K, N) \gamma]^N \leq c_2(K, N) \gamma^N,$$

where $c_2(K, N) = M_N K (1 + c_1(K, N))^N$, and we note that \mathcal{P}_x^{-1} is a polynomial of degree at most $d \leq N$. Combining this with (5.29) and K^{N-1} estimate from (5.22) for the last term in (5.27), we obtain the following estimate for the norm of J_x in (5.26):

$$(5.31) \quad \|J_x\| \leq c(N) c_2(K, N) \gamma^N \cdot r \|t\|^\alpha \|\bar{S}\|_c \cdot K^{N-1} \leq c_3(K, N) \gamma^N \cdot r \|t\|^\alpha \cdot \|\bar{S}\|_c.$$

where $c_3(K, N) = c(N) c_2(K, N) K^{N-1}$ and $c(N)$ is an estimate on the number of terms in J_x .

Now we estimate the main term in (5.26) using (5.15), (5.21), (5.30), and (5.28):

$$(5.32) \quad \begin{aligned} & \|D_{t''}^{(1)} (\mathcal{P}_x)^{-1} \circ D_{t'}^{(N)} S_{f_x} \circ D_t^{(1)} \mathcal{F}_x\| \leq \\ & \leq \|D_{t''}^{(1)} (\mathcal{P}_x)^{-1}\| \cdot \|D^{(N)} S_{f_x}\|_\alpha \|t'\|^\alpha \cdot \|D_t^{(1)} \mathcal{F}_x\|^N \leq \\ & \leq e^{-\chi_1 + \varepsilon + \varepsilon'} \cdot \|\bar{S}\|_c \cdot e^{\alpha(\chi_\ell + \varepsilon + \varepsilon')} \|t\|^\alpha \cdot e^{N(\chi_\ell + \varepsilon + \varepsilon')} = e^{-\delta} \|t\|^\alpha \|\bar{S}\|_c, \end{aligned}$$

by the definition of δ (5.19) where $\nu = -(N + \alpha)\chi_\ell + \chi_1$. By (5.23) we have $e^{-\delta} = 1 - 2\theta$. Finally we estimate (5.26) combining (5.31) and (5.32). For any $\bar{R} \in B_\gamma$

$$\|D_t^{(N)} ([D_{\bar{R}} T] \bar{S})_x\| \leq \|t\|^\alpha \cdot \|\bar{S}\|_c (1 - 2\theta + c_3(K, N) \gamma^N \cdot r) \leq \|t\|^\alpha \cdot \|\bar{S}\|_c (1 - \theta).$$

since $r \leq \theta/(c_3(K, N) \gamma^N)$ by (5.24). Then for all $\bar{R} \in B_\gamma$ we obtain

$$\begin{aligned} \|D_t^{(N)}([D_{\bar{R}}T]\bar{S})_x\| &\leq \|t\|^\alpha \cdot \|\bar{S}\|_{\mathcal{C}}(1 - \theta), \quad \text{hence} \\ \|D^{(N)}([D_{\bar{R}}T]\bar{S})_x\|_\alpha &\leq (1 - \theta) \cdot \|\bar{S}\|_{\mathcal{C}}, \quad \text{and so} \\ \|[D_{\bar{R}}T]\bar{S}\|_{\mathcal{C}} &= \sup_x \|D^{(N)}(T(\bar{S}))_x\|_\alpha \leq (1 - \theta) \cdot \|\bar{S}\|_{\mathcal{C}}. \end{aligned}$$

Thus $\|D_{\bar{R}}T\| \leq 1 - \theta$ for all $\bar{R} \in B_\gamma$. Since $\|T(\bar{0})\|_{\mathcal{C}} \leq \theta\gamma$ from the definition of γ (5.23), the operator T is a contraction from B_γ to itself. Thus T has a unique fixed point $\bar{H} \in B_\gamma$, which is section of \mathcal{C} bounded in $C^{N,\alpha}$ norm and continuous in C^N norm. The corresponding family of $C^{N,\alpha}$ maps $\mathcal{H}_x = \text{Id} + H_x$ satisfies (5.13), i.e. conjugates \mathcal{P}_x and \mathcal{F}_x . Then the maps \mathcal{H}_x defined on $B_{x,r}$ can be uniquely extended to $C^{N,\alpha}$ diffeomorphisms on $B_{x,\rho}$, and then on $B_{x,\sigma}$, by the invariance

$$\mathcal{H}_x(t) = (\mathcal{P}_x^k)^{-1} \circ \mathcal{H}_{f^k x} \circ \mathcal{F}_x^k(t)$$

since for each $t \in B_{x,\sigma}$ we have $\mathcal{F}_x^k(t) \in B_{x,r}$ for some k .

5.3. Prove of part (2): the (non)uniqueness of \mathcal{H} and \mathcal{P} . This essentially follows from the ‘‘uniqueness’’ of the construction. First, starting with $\mathcal{H}_1 = \tilde{\mathcal{H}}$ we inductively construct coordinate changes $\mathcal{H}_k = \{\mathcal{H}_{k,x}\}$ for $k = 1, \dots, N$ so that the corresponding normal form $\mathcal{P}_{k,x}$ is of sub-resonance type and their Taylor polynomials $\mathcal{H}_{k,x}(t) \sim \sum_{n=1}^N H_{k,x}^{(n)}(t)$ coincide with the Taylor polynomial of \mathcal{H} to order k , that is $H_x^{(n)} = H_{k,x}^{(n)}$ for $n = 1, \dots, k$. For $\mathcal{H}_1 = \tilde{\mathcal{H}}$ we have $H_x^{(1)} = H_{1,x}^{(1)} = \text{Id}$ and $\mathcal{P}_{1,x}$ is sub-resonance by the assumption.

Suppose \mathcal{H}_{k-1} , $k \geq 2$, is constructed and has $H_x^{(n)} = H_{k-1,x}^{(n)}$ for $n = 1, \dots, k-1$. Then \mathcal{P} and \mathcal{P}_{k-1} have the same terms up to order $k-1$. Hence $H_{k-1,x}^{(k)}$ and $H_x^{(k)}$ satisfy the same equation (5.4) when projected to the factor-bundle $\mathcal{Q}^{(k)}/\mathcal{S}^{(k)}$. Indeed, the Q term defined by (5.3) is composed only of $F^{(i)}$ and terms $H^{(i)}$ and $P^{(i)}$ with $1 < i \leq k-1$, which are the same for \mathcal{H}_{k-1} and \mathcal{H} . By uniqueness we obtain that

$$H_x^{(k)} = H_{k-1,x}^{(k)} + \Delta_x^{(k)}, \quad \text{where } \Delta_x^{(k)} \in \mathcal{S}_x^{(k)}.$$

Then the coordinate change $\mathcal{H}_{k,x} = (\text{Id} + \Delta_x^{(k)}) \circ \mathcal{H}_{k-1,x}$ has the same Taylor terms as \mathcal{H} up to order k and, since the polynomial $\text{Id} + \Delta_x^{(k)}$ is in \mathcal{S}_x , \mathcal{H}_k conjugates \mathcal{F} to a sub-resonance normal form $\mathcal{P}_{k,x} = (\text{Id} + \Delta_{f^k x}^{(k)}) \circ \mathcal{P}_{k-1,x} \circ (\text{Id} + \Delta_x^{(k)})^{-1}$.

Thus in N steps we obtain the coordinate change

$$\mathcal{H}_{N,x} = G_x \circ \tilde{\mathcal{H}}_x, \quad \text{where } G_x = (\text{Id} + \Delta_x^{(N)}) \circ \dots \circ (\text{Id} + \Delta_x^{(2)}) \in \mathcal{S}_x,$$

which has the same Taylor terms at 0 as \mathcal{H} up to order N . In fact, for $n > d$ we have $\mathcal{S}^{(n)} = 0$ and hence $\Delta^{(n)} = 0$, so that $\mathcal{H}_N = \mathcal{H}_d$.

Finally, we conclude $\mathcal{H}_d = \mathcal{H}_N = \mathcal{H}$ by the ‘‘moreover part of (2), which follows from the uniqueness of the fixed point in the last step of the construction.

5.4. Prove of part (1'): construction of resonance normal form.

Now we construct a *polynomial* coordinate change \mathcal{H} that brings the sub-resonance normal form $\mathcal{P}_x = \sum_{n=1}^d P_x^{(n)}$ to a resonance normal form $\tilde{\mathcal{P}}_x = \sum_{n=1}^d \tilde{P}_x^{(n)}$.

We will inductively construct the terms of polynomial coordinate changes $\mathcal{H}_x = \sum_{n=1}^d H_x^{(n)}$. All terms will be in the group \mathcal{S} of sub-resonance polynomials, so the process will stop in d steps and yield sub-resonance of polynomial diffeomorphisms \mathcal{H}_x and resonance normal form $\tilde{\mathcal{P}}_x$. The base case is $n = 1$, where we take $H_x^{(1)} = \text{Id}$, which leaves $\tilde{P}_x^{(1)} = P_x^{(1)} = F_x$ in the resonance form, i.e. block triangular.

Now we assume inductively that the terms of degree $k < n$ are constructed so that $H^{(k)}$ is a continuous section of $\mathcal{S}^{(k)}$ and $\tilde{P}_x^{(k)}$ is a resonance polynomial and is continuous in x . As before, we consider the terms of degree n in the conjugacy equation $\mathcal{H}_{fx} \circ \mathcal{P}_x = \tilde{\mathcal{P}}_x \circ \mathcal{H}_x$

$$P_x^{(n)} + H_{fx}^{(n)} \circ F_x + \sum H_{fx}^{(i)} \circ P_x^{(j)} = F_x \circ H_x^{(n)} + \tilde{P}_x^{(n)} + \sum \tilde{P}_x^{(j)} \circ H_x^{(i)},$$

where the summations are over all i and j such that $ij = n$ and $1 < i, j < n$. Then

$$(5.33) \quad \tilde{P}_x^{(n)} \circ F_x^{-1} = -F_x \circ H_x^{(n)} \circ F_x^{-1} + H_{fx}^{(n)} + Q_x, \quad \text{where}$$

$$Q_x = \left(P_x^{(n)} + \sum_{ij=n, 1 < i, j < n} H_{fx}^{(i)} \circ P_x^{(j)} - \tilde{P}_x^{(j)} \circ H_x^{(i)} \right) F_x^{-1}.$$

We note that Q_x is composed only of terms $H^{(i)}$ and $\tilde{P}^{(i)}$ with $1 < i < n$, which are already constructed, and terms $P^{(i)}$ with $1 < i \leq n$, which are given. Since composition of sub-resonance polynomials is a sub-resonance polynomial, by the inductive assumption Q_x is a continuous section of $\mathcal{S}^{(n)}$.

We consider splitting $\mathcal{S}_x^{(n)} = \mathcal{R}_x^{(n)} \oplus \mathcal{S}\mathcal{S}_x^{(n)}$, where $\mathcal{R}_x^{(n)}$ and $\mathcal{S}\mathcal{S}_x^{(n)}$ denote the maps of resonance and strict sub-resonance types respectively. We seek $H_x^{(n)}$ so that the right side of (5.33) is in $\mathcal{R}_x^{(n)}$, and hence so will be $\tilde{P}_x^{(n)}$ when defined by this equation.

Projecting (5.33) to the factor bundle $\mathcal{S}^{(n)}/\mathcal{R}^{(n)}$ we need to solve the equation

$$(5.34) \quad 0 = -F_x \circ \bar{H}_x^{(n)} \circ F_x^{-1} + \bar{H}_{fx}^{(n)} + \bar{Q}_x,$$

where $\bar{H}^{(n)}$ and \bar{Q} are the projections of $H^{(n)}$ and Q respectively. We consider the automorphism Φ^{-1} of the bundle $\mathcal{S}^{(n)}$ covering f with fiber maps

$$(5.35) \quad \Phi_x^{-1} : \mathcal{S}_x^{(n)} \rightarrow \mathcal{S}_{fx}^{(n)} \quad \text{where} \quad \Phi_x^{-1}(R) = F_x \circ R \circ F_x^{-1}.$$

Since F preserves the splitting $\mathcal{E} = \mathcal{E}^1 \oplus \dots \oplus \mathcal{E}^\ell$, the resonance and strict sub-resonance types are preserved by Φ^{-1} . We denote by $\bar{\Phi}^{-1}$ the induced automorphism of $\mathcal{S}^{(n)}/\mathcal{R}^{(n)}$ and see that (5.34) becomes

$$(5.36) \quad \bar{H}_{fx}^{(n)} = \tilde{\Phi}_x^{-1}(\bar{H}_x^{(n)}), \quad \text{where} \quad \tilde{\Phi}_x^{-1}(R) = \bar{\Phi}_x^{-1}(R) - \bar{Q}_x.$$

Thus a solution of (5.34) is a $\tilde{\Phi}^{-1}$ -invariant section of $\mathcal{S}^{(n)}/\mathcal{R}^{(n)}$.

Lemma 5.5. *The map $\Phi^{-1} : \mathcal{SS}^{(n)} \rightarrow \mathcal{SS}^{(n)}$ given by (5.35) is a contraction over f , and hence so is $\tilde{\Phi}^{-1} : \mathcal{S}^{(n)}/\mathcal{R}^{(n)} \rightarrow \mathcal{S}^{(n)}/\mathcal{R}^{(n)}$ given by (5.36). More precisely, $\|\Phi_x^{-1}(R)\| \leq e^{\lambda+(d+2)\varepsilon} \cdot \|R\|$.*

Proof. The statement about $\tilde{\Phi}^{-1}$ follows since the linear part $\bar{\Phi}^{-1}$ of $\tilde{\Phi}^{-1}$ is given by Φ^{-1} when $\mathcal{S}^{(n)}/\mathcal{R}^{(n)}$ is naturally identified with $\mathcal{SS}^{(n)}$.

By Lemma 5.1, for polynomials of homogeneous type $s = (s_1, \dots, s_\ell)$ with $s_1 + \dots + s_\ell = n$ we have $\|\Phi_x^{-1}(R)\| \leq e^{\chi_i - \sum s_j \chi_j + (n+1)\varepsilon} \|R\|$. For all strict sub-resonance homogeneous types we have $\chi_i - \sum_{j=1}^{\ell} s_j \chi_j \leq \mu$ by the definition of μ (3.12) and hence for any $R \in \mathcal{SS}_x^{(n)}$ we have $\|\Phi_x(R)\| \leq e^{\mu+(n+1)\varepsilon} \cdot \|R\|$. Since $n \leq d$ the exponent satisfies $\mu + (n+1)\varepsilon \leq \lambda + (d+1)\varepsilon < 0$ since $\varepsilon < \varepsilon_0$ given by (3.13). \square

We conclude that $\tilde{\Phi}^{-1}$ is a contraction and hence has a unique continuous invariant section $\bar{H}^{(n)}$. We choose a continuous section $H^{(n)}$ of $\mathcal{S}^{(n)}$ which projects to $\bar{H}^{(n)}$, which is defined uniquely up to a section of $\mathcal{R}^{(n)}$. For example one can take $H^{(n)}$ in $\mathcal{SS}^{(n)}$. Once $H^{(n)}$ is chosen, we define $\tilde{P}_x^{(n)}$ by equation (5.33) and get a continuous section $\tilde{P}^{(n)}$ of $\mathcal{R}^{(n)}$. This completes the inductive step and the construction of \mathcal{H} and $\tilde{\mathcal{P}}$.

5.5. Prove of part (2'): the (non)uniqueness for resonance normal form.

This follows from the ‘‘uniqueness’’ of the construction in the previous section similarly to the proof of part (2). The process of transition from $\tilde{\mathcal{H}}'_x$ to \mathcal{H}'_x stays in the group of resonance polynomials \mathcal{R}_x and we obtain $\mathcal{H}_x = \mathcal{H}_{d,x} = G_x \circ \tilde{\mathcal{H}}'_x$, where $G_x \in \mathcal{R}_x$.

5.6. Proof of part (3): centralizer. First we prove that the derivative of \mathcal{G} at zero section, $\Gamma_x = D_0\mathcal{G}_x$, is sub-resonance. Since Γ_x is linear, this is equivalent to the fact that Γ_x preserves the flag of fast sub-bundles associated with the splitting (3.10). Suppose to the contrary that for some $x \in X$ and some $i > j$ we have a unit vector t in \mathcal{E}_x^j such that $t' = \Gamma_x(t)$ has nonzero component $t'_i \neq 0$ in \mathcal{E}_{gx}^i . Then we have

$$\|(F_{gx}^n \circ \Gamma_x)(t)\| \geq \|F_{gx}^n(t'_i)\| \geq e^{(\chi_i - \varepsilon)n} \|t'_i\|.$$

On the other hand, since the extensions and hence their derivatives commute, we have

$$\|(F_{gx}^n \circ \Gamma_x)(t)\| = \|\Gamma_{f^n x}(F_x^n(t))\| \leq \|\Gamma_{f^n x}\| \cdot e^{(\chi_j + \varepsilon)n} \|t\| \leq C e^{(\chi_j + \varepsilon)n},$$

which is impossible for large n as ε is small enough. Indeed, since $i > j$ we have non sub-resonance relation $-\chi_i + \chi_j < 0$, so by definition (3.11) of $\tilde{\lambda}$ we have $-\chi_i + \chi_j \leq \tilde{\lambda} < 0$, and hence by definition (3.13) of ε_0 we have $\varepsilon_0 \leq -\tilde{\lambda}/(d+2) \leq -\tilde{\lambda}/3 < (\chi_i - \chi_j)/2$. Since $\varepsilon < \varepsilon_0$, this yields $\chi_j + \varepsilon < \chi_i - \varepsilon$.

Similarly, we can further show that Γ_x is resonance, i.e. preserves the splitting. Using notations above, if $i < j$ we can estimate backward iterates: for $n < 0$ we have

$$e^{(\chi_i + \varepsilon)n} \|t'_i\| \leq \|F_{gx}^n(t'_i)\| \leq \|(F_{gx}^n \circ \Gamma_x)(t)\| = \|\Gamma_{f^n x}(F_x^n(t))\| \leq \|\Gamma_{f^n x}\| \cdot e^{(\chi_j - \varepsilon)n} \|t\|,$$

which is impossible since $\chi_i + \varepsilon < \chi_j - \varepsilon$. This follows as above from (3.13) and (3.12) since for a strict sub-resonance $\chi_i < \chi_j$ we have $\chi_i - \chi_j \leq \mu < 0$ and hence $\varepsilon < \varepsilon_0 \leq -\mu/(d+1) \leq -\mu/2 < (-\chi_i + \chi_j)/2$.

Now we consider a new family of coordinate changes

$$\tilde{\mathcal{H}}_x = \Gamma_x^{-1} \circ \mathcal{H}_{gx} \circ \mathcal{G}_x$$

which also satisfies $\tilde{\mathcal{H}}_x(0) = 0$ and $D_0\tilde{\mathcal{H}}_x = \text{Id}$. A direct calculation shows that

$$\begin{aligned} \tilde{\mathcal{H}}_{fx} \circ \mathcal{F}_x \circ \tilde{\mathcal{H}}_x^{-1} &= \Gamma_{fx}^{-1} \circ \mathcal{H}_{fgx} \circ \mathcal{G}_{fx} \circ \mathcal{F}_x \circ \mathcal{G}_x^{-1} \circ \mathcal{H}_{gx}^{-1} \circ \Gamma_x = \\ &= \Gamma_{fx}^{-1} \circ \mathcal{H}_{fgx} \circ \mathcal{F}_{gx} \circ \mathcal{H}_{gx}^{-1} \circ \Gamma_x = \Gamma_{fx}^{-1} \circ \mathcal{P}_{gx} \circ \Gamma_x = \tilde{\mathcal{P}}_x. \end{aligned}$$

Hence if \mathcal{P}_x is a sub-resonance polynomial then so is $\tilde{\mathcal{P}}_x$ as a composition of sub-resonance polynomials. Now part (2) of the theorem gives $\tilde{\mathcal{H}}_x = G_x \mathcal{H}_x$ for some $G_x \in \mathcal{S}_x$ which depends continuously on x . Then the definition of $\tilde{\mathcal{H}}_x$ yields

$$\mathcal{H}_{gx} \circ \mathcal{G}_x = \Gamma_x \circ \tilde{\mathcal{H}}_x = (\Gamma_x G_x) \circ \mathcal{H}_x$$

so that $\mathcal{H}_{gx} \circ \mathcal{G}_x \circ \mathcal{H}_x^{-1} = \Gamma_x G_x$, which is again a sub-resonance polynomial, as claimed.

Similarly, using part (2'), we can obtain that if \mathcal{P}_x , and hence $\tilde{\mathcal{P}}_x$, are resonance polynomials then $\mathcal{H}_{gx} \circ \mathcal{G}_x \circ \mathcal{H}_x^{-1} = \Gamma_x G_x$, where G_x , and hence $\Gamma_x G_x$, are also resonance polynomials for each $x \in X$.

This completes the proof of Theorem 4.3. \square

5.7. Proof of Corollary 4.4. By part (2) of Theorem 4.3, if we fix a choice of Taylor polynomials of degree d for \mathcal{H}_x , then the family \mathcal{H}_x is unique. Then for each $N > d$ we can do the construction in part (1) with this fixed choice of Taylor polynomials and obtain the family of C^N diffeomorphisms \mathcal{H}_x . By uniqueness, all these families coincide and hence \mathcal{H}_x are C^∞ diffeomorphisms. For $\tilde{\mathcal{H}}$ in part (1') the smoothness follows since, as we will show, it is a composition of \mathcal{H} with a polynomial diffeomorphism.

6. PROOF OF THEOREM 4.6 AND COROLLARY 4.8

The parts (1), (1'), (2), (2'), (3), and (5) of Theorem 4.6 are obtained using Theorem 4.3 as follows. We consider the vector bundle $\mathcal{E} = T\mathcal{W}$ with $\mathcal{E}_x = T_x\mathcal{W}$. To construct extension \mathcal{F} as in Theorem 4.3 we restrict f to the leaves of \mathcal{W} and obtain \mathcal{F}_x by identifying $B(x, \sigma) \subset T_x\mathcal{W}$ with a neighborhood of x in \mathcal{W}_x using exponential map. It is easy to see that Assumptions 4.1 are satisfied with $N = \lfloor r \rfloor$ and $\alpha = r - \lfloor r \rfloor$.

Hence Theorem 4.3 gives existence of families $\{\mathcal{H}_x\}_{x \in X}$ and $\{\mathcal{H}'_x\}_{x \in X}$ of local normal form coordinates as in (1) and (1') satisfying "uniqueness properties" (2) and (2'). Then, as in Remark 4.5, they can be extended uniquely to global diffeomorphisms $\mathcal{H}_x : \mathcal{W}_x \rightarrow \mathcal{E}_x$. We note that the Hölder condition at 0 in Theorem 4.3 implies that \mathcal{H}_x in (1) is globally Hölder along \mathcal{W}_x by part (4). This also implies that \mathcal{H}'_x in (1') is globally Hölder since by (2) it differs from \mathcal{H}_x by a polynomial diffeomorphism.

To prove (3), we similarly restrict g to the leaves of \mathcal{W} and obtain the extension \mathcal{G} commuting with \mathcal{F} , so that the result follows from (3) of Theorem 4.3.

The existence of $\{\mathcal{H}_x\}_{x \in X}$ as in part (5) can be obtained by constructing the Taylor terms of \mathcal{H}_x that depend smoothly on x as indicated in Remark 5.3, see [KS16] for more details of this argument.

Part (4) requires a different argument for which we refer to [KS16, KS17]. First one shows inductively that the Taylor polynomial of the transition maps is sub-resonance, and then argues that error term is zero.

The first part of Corollary 4.8 follows directly since the splitting (3.4) is trivial, $d = 1$, and hence there are no non-linear sub-resonance polynomials. This also means that λ and μ are not deduced as there are no corresponding relations and we get $\varepsilon_0 = \lambda/3 = -\chi/3$. Hence we need $\varepsilon < \min\{\varepsilon_0/3, \nu/(N + \alpha + 1)\}$. The second term is $-\chi(N + \alpha - 1)/(N + \alpha + 1)$ and smallest for $N = 1$ and gives $-\chi\alpha/(2 + \alpha)$, which is less than $\varepsilon_0/3$. This means that we need $\varepsilon < -\chi\alpha/(2 + \alpha)$, which yields the interval $(-\chi(1 + \frac{\alpha}{2+\alpha}), -\chi(1 - \frac{\alpha}{2+\alpha}))$ with endpoint ratio $1 + \alpha$.

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DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802, USA.

E-mail address: kalinin@psu.edu