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A REMARK ON NEWMAN'S THEOREM[†)]

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1. The main results.

1.1. Newman's well-known theorem [1] reads:

Let M be a topological manifold endowed with a metric d. Then there exists $\varepsilon = \varepsilon(M,d) > 0$ such that an arbitrary action of a finite group G by homeomorphisms on M is trivial, provided that $d(\tau x, x) < \varepsilon$ for all $\tau \in G$ and $x \in M$.

A number of authors generalized the theorem and simplified its proof (see [2–5]). The most essential generalization belongs to A. V. Chernavskiĭ [3], who proved an analogous theorem under the assumption that, instead of the action of a finite group, there is a continuous partition of M induced by a finite-to-one, open and closed mapping. In line with [6], by a *pseudosubmersion* we will mean a continuous, surjective, open and closed mapping $f : X \to Y$ between topological spaces which is not a homeomorphism and which has $\operatorname{card} f^{-1}(y) < \infty$ for all $y \in Y$. It was in particular proven in [3] that, for a topological manifold endowed with a metric, the Newman constant to be defined below is always positive.

1.2. DEFINITION. The *Newman constant* $\varepsilon(X,d)$ of a connected metric space (X,d) is defined as the least $\varepsilon > 0$ for which there is a pseudosubmersion $F : X \to Y$ with $\operatorname{diam} f^{-1}(y) < \varepsilon$ for all $y \in Y$.

The above-mentioned results related only to the case of topological manifolds, except the article [2] in which more general spaces were considered in the case of the action of a finite group. However, strong requirements were imposed on the local homological structure of the spaces; in consequence, such spaces as, say, finite trees fall out of the consideration.

Our purpose is to propose a generalization of Newman's theorem which would cover such geometrically interesting objects as the Alexandrov spaces with curvatures bounded below (or above) and would make it possible to obtain geometric estimates for the Newman constant.

The main result is the following

1.3. Theorem. *Let a connected locally compact metric space (X,d) be representable as the closure of the union of finitely many open subsets A_i which are topological manifolds (possibly of different dimensions). Then $\varepsilon(X,d) > 0$.*

(In the article [7] there are some estimates for ε in dependence on the injectivity radius and curvatures. If we confine ourselves only to the isometry group then the corresponding "Newman constant" admits of stronger estimates (see [8]).)

By virtue of the stratification theorem (see [9] or [10, Theorem 13.2]), the conditions of Theorem 1.3 hold for compact finite-dimensional Alexandrov spaces with curvature bounded below; hence, for such spaces we have $\varepsilon(X,d) > 0$.

Denote by $\mathfrak{M}(K,n,D,V)$ the class of compact Alexandrov spaces M of dimension n whose curvature is at least K and for which $\operatorname{diam} M \le D$ and $\operatorname{Vol} M \ge V$. (The presence of the boundary is not excluded for M.)

Theorem 1.3, the compactness theorem for Alexandrov spaces [10, Theorem 8.5], and Perelman's stability theorem [10, Theorem 13.2; 11] imply

1.4. Corollary. *For positive D and V, an arbitrary K, and an integer n, there exists $\varepsilon_0 = \varepsilon_0(K,n,D,V) > 0$ such that $\varepsilon(M) \ge \varepsilon_0$ for all $M \in \mathfrak{M}(K,n,D,V)$.*

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The corollary generalizes one result of [6] proven for closed Riemannian manifolds.

The conditions of Theorem 1.3 hold for some Alexandrov spaces of nonpositive (or bounded above) curvature which have, roughly speaking, "finitely many ramifications"; for example, for finite trees. At the same time, the example of the tree with arbitrarily many short twin branches shows that in the general case the conclusion of the theorem fails for such spaces.

1.5. Despite the formal novelty of Theorem 1.3, its proof can be achieved by directly generalizing A. V. Chernavskiĭ's arguments. However, here we give a simpler proof by combining the ideas by A. V. Chernavskiĭ [3] and A. Dress [4]. The article [4] relates to the case of the action of a finite group. In the more general situation of a continuous partition of a manifold, Dress's method was applied in the article [5] but under the more rigid constraint that the quotient space is a manifold without boundary; in consequence, even the case of the action of a finite group is not completely covered (see also Remark 3.2).

We rely upon Theorem 2 of [3] whose proof involves rather intricate constructions. However, they can be avoided if one makes use of the method of the article [4] as it is done in the arguments below.

2. Preliminaries and notation.

2.1. Let a continuous surjective mapping $f : X \to Y$ between topological spaces be open and closed simultaneously. Then the corresponding partition of X into the inverse images of points ("partition elements") is called *continuous*. Clearly, the quotient space is canonically homeomorphic to Y. A set $A \subset X$ is called *saturated* if $f^{-1}(f(A)) = A$. The *multiplicity* of a partition element is the number of points in it.

2.2. The following facts are easily verifiable and probably well-known although we find it difficult to indicate precise references to their proofs.

If X is locally connected and locally compact and the inverse images of points are compact, then the inverse image in X of even open connected subset $A \subset Y$ consists of finitely many components and each of the components is mapped onto the whole of A. In particular, if a partition element contains finitely many points and if a connected neighborhood of the image of the element in Y is sufficiently small then the inverse image of the neighborhood has as many components as many points are in the element; moreover, the closures of the components are disjoint. A continuous partition is induced on each component.

Let X be a connected topological manifold and let the multiplicity of all partition elements be finite. By virtue of [3, Theorem 1], the multiplicity of the partition is bounded; i.e., there is a positive integer m such that the multiplicity of the partition elements does not exceed m.

2.3. REMARK. The following generalization of Theorem 1 of [3] holds: *Let X have the same form as in Theorem* 1.3 *and let $f : X \to Y$ be a pseudosubmersion. Then the multiplicity of the partition induced on X is bounded.*

The proof can be translated from [3] practically without changes.

2.4. Introduce the notation: K_i is the union of partition elements containing at most i points; $\Phi_i = K_i \setminus K_{i-1}$ (assume $K_0 = \varnothing$). By continuity of the partition, the sets K_i are closed, while Φ_i are open in K_i and locally compact. On each Φ_i the mapping is locally homeomorphic and has multiplicity i. Clearly, all the K_i's and Φ_i's are saturated.

Theorem 2 of [3] asserts that if a continuous partition of bounded multiplicity is given on a connected manifold, then the partition elements of maximal multiplicity form an everywhere dense open set. Thus, K_i is nowhere dense if i is less than the maximal multiplicity m. Moreover, Φ_i with $i > m$ may locally divide the manifold only if m is even (for $i > 1$ this is proven in [3, § 7]; the case $i = 1$ is settled similarly on use made of Theorem 2 of [3]). Also, observe that $\dim f(K_{m-1}) < n$, where n is the dimension of the manifold. Indeed, since $\dim \Phi_i < n$ for $i < m$ (because Φ_i has no interior points (see [12, p. 69])) and f is locally homeomorphic on Φ_i, it follows that $\dim f(\Phi_i) < n$. Since f is closed, $f(K_i)$ are closed and, because of $\Phi_1 = K_1$, we successively obtain $\dim f(K_i) < n$ for $i < m$ (see [12, p. 55]).

3. The main lemma. To prove Theorem 1.3, we first establish an analog of Lemma 3 of [4] in the case of continuous partitions.

3.1. Lemma. *Let $f : U \to V$ be a pseudosubmersion of a bounded open connected set $U \subset \mathbb{R}^n$ into a topological space V. Then*

$$D \equiv \max\{\min\{\|x - z\| : z \in \partial U\} : x \in U\} \leq C \equiv \sup\{\operatorname{diam} f^{-1}(y) : y \in V\}.$$

3.2. Remark. Lemma 3.1 was formulated and used in [7] with a reference to a preliminary version of the article [5]. However, in the very article [5] a considerably weaker assertion is proven (it is assumed that V is a manifold without boundary). Probably, the misunderstanding is caused by the fact that the authors of [7] were acquainted only with a preliminary version of [5].

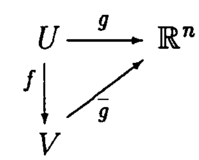
3.3. Proof of Lemma 3.1. Let m stand for the maximal multiplicity of the partition. Construct a special mapping $g : U \to \mathbb{R}^n$ that associates the center of mass with the points of each partition element. To this end, put $g(x) = \frac{1}{m}\sum x_i$ for $x \in \Phi_m$. Here and below, the summation is assumed to be taken over all points x_i of a partition element. For all $x \in U$, define the ramification number $w(x)$ as the maximal multiplicity of the partition induced on a sufficiently small neighborhood about x. Then $\sum w(x_i) = m$.

Verify soundness of the definition. If a neighborhood O of a point $y \in V$ is sufficiently small, then $f^{-1}(O) = \cup W_i$ is a partition mentioned in Section 2.2. A continuous partition is induced on every W_i. Denote by S_i the union of the elements of maximal multiplicity of the partition. Since S_i is open and dense in W_i and since Φ_m is open and dense in U, it follows that $R_i \equiv S_i \cap \Phi_m$ too is open and dense in W_i. It is clear that $\bigcap_i f(R_i) \neq \varnothing$, and consequently the sum of the maximal multiplicities of the partitions on W_i equals m.

Now, we in a unique fashion extend the function g to the whole of U by continuity:

$$g(x) = \frac{1}{m}\sum w(x_i)x_i \quad \text{for all} \quad x \in U.$$

Thus, the mapping g is continuous and is constant on the partition elements; i.e., it admits the commutative diagram



(*)

Let $B(x, R) = \{z \in \mathbb{R}^n : \|x - z\| < R\}$ be an open ball. Suppose that Lemma 3.1 is false. Then $D > C$ and there exists $x_0 \in \Phi_m$ (since Φ_m is everywhere dense) such that $B(x_0, C + \varepsilon) \subset U$. Demonstrate that the above-defined function g can be redefined so that $g(x_0) = x_0$. Indeed, let $f^{-1}f(x_0) = \{x_0, x_1, \dots, x_{m-1}\}$ and let the balls $B(x_j, \delta) \subset \Phi_m$ be pairwise disjoint. Put $\tilde{h}(x) = \delta^{-1}\|x - x_j\|$ for $x \in B(x_j, \delta)$, $j \neq 0$; $\tilde{h}(x) = 1$ for $x \in B(x_0, \delta)$; and $h(x) = \tilde{h}(x)(\sum \tilde{h}(x_i))^{-1}$ for $x \in \cup B(x_j, \delta)$. It is now clear that the function g_1 defined as $g_1(x) = \sum h(x_i)x_i$ for $x \in \cup B(x_i, \delta)$ and as $g_1(x) = g(x)$ at the other points is continuous and admits a commutative diagram analogous to diagram (*) too; moreover, $g_1(x_0) = x_0$.

Define a homotopy $H : U \times I \to \mathbb{R}^n$ by the equality $H(x, t) = tx + (1 - t)g_1(x)$. We have

$$x_0 = H(x_0, I) \notin H(U \setminus B(x_0, C + \varepsilon/2), I),$$

for otherwise $x_0 = tx + (1-t)g_1(x)$ for some $x \notin B(x_0, C+\varepsilon/2)$ and $C < \|x_0 - x\| \leq (1-t)\sum h(x_i)\|x_i - x\| \leq C$. Consequently, $H^{-1}(x_0)$ is closed in $\mathbb{R}^n \times I$ and is thus compact. Now, we can assert that $\deg_{x_0} g_1 = 1$ for every ring of coefficients. Observe that $\deg_y f = 0$ for all $y \in f(\Phi_m) \subset V$ and for the coefficient ring $\mathbb{Z}_p$, where p is a minimal prime divisor of m, since f on Φ_m is a covering and either Φ_m is connected or m is odd. If we known that $x_0 = g_1(x_0) \notin g(K_{m-1})$, we would have the equality $\deg_{x_0} g_1 = 0$ for the ring $\mathbb{Z}_p$ and thereby arrive at a contradiction proving the lemma.

The closed set $N = f(K_{m-1})$ has dimension less than n; therefore (see [12, p. 108]), the point $x_0 \in \mathbb{R}^n$ is an unstable value of the mapping $\bar{g}_1|_N : N \to \mathbb{R}^n$; i.e., there are arbitrarily close mappings (in the uniform metric) from N into $\mathbb{R}^n$ not containing x_0 in the range. By normality (metrizability) of V, any such mapping can be extended to V, with the value at the point $v_0 = f(x_0)$ preserved and the distance to $\bar{g}_1$ not increased. Take a mapping $\bar{g}_2 : V \to \mathbb{R}^n$ such that, for the corresponding mapping $g_2 = g$ in a diagram of the form (*), the homotopy $H_2(x,t) = xt + (1-t)g_2(x)$ possesses the property $H_2^{-1}(x_0) \subset B(x_0, C + \varepsilon/4)$. Then $H_2^{-1}(x_0)$ is compact and $\deg_{x_0} g_2 = 1$ for every ring of coefficients. However, since g_2 remained constant on the partition elements and $x_0 = g_2(x_0) \notin g_2(K_{m-1})$, we have $\deg_{x_0} g_2 = 0$ for the coefficient ring $\mathbb{Z}_p$, which proves the lemma.

4. Proof of Theorem 1.3. Denote by F the union of the partition elements that are singletons. The set F is closed. For each manifold A_i the following alternative holds: either $A_i \subset F$ or $A_i \cap F$ is nowhere dense in A_i. Indeed, if we had $\operatorname{int}(A_i \cap F) \neq \varnothing$ and $A_i \not\subset F$, then we would take a saturated neighborhood U of a point on the boundary of $\operatorname{int}(A_i \cap F)$ such that $U \subset A_i$. But then on the manifold U we would have a nontrivial continuous partition for which the set of singleton elements has a nonempty interior, contradicting [3, Theorem 2].

Take a chart $\varphi_i : \operatorname{Cl} B(0,2) \to A_i$ in each A_i. There exists $\varepsilon_i > 0$ enjoying the properties:

(1) if x', $z' \in \varphi_i(\operatorname{Cl} B(0,2))$ and $d(x', z') < \varepsilon_i$ then

$$\left\|\varphi_i^{-1}(x') - \varphi_i^{-1}(z')\right\| < 1;$$

(2) if $d(x, \varphi_i(B(0,1))) < \varepsilon_i$ then $x \in \varphi_i(B(0,2))$.

Put $\varepsilon = \min \varepsilon_i$ over i. Now, assume $\operatorname{diam} f^{-1}(y) < \varepsilon$ for all $y \in Y$. If the partition is nontrivial then $A_{i_0} \not\subset F$ for some i_0 and, consequently, $A_{i_0} \cap F$ is nowhere dense in A_{i_0}. But then the partition induced on the set

$$U = \left(\varphi_{i_0}^{-1} \circ f^{-1} \circ f \circ \varphi_{i_0}\right)(B(0,1)) \subset B(0,2)$$

is nontrivial, contradicting Lemma 3.1.

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